MATH 111: Introduction to Calculus | Winter 2026

Deewang Bhamidipati, Carleton College

Contents

Day 1 (1.1, 1.2, 1.3) 2

Day 2 (1.3, 1.4, 1.5) 10

Day 3 (2.1, 2.2) 18

Day 4 (2.2, 2.3) 25

Day 5 (2.5, 2.6) 32

Day 6 (2.7, 2.8) 41

Day 7 (3.1, 3.2) 47

Day 8 (3.2, 3.3) 55

Day 9 (3.4) 62

Day 10 (3.5) 69

Days 11 & 12 – Review & Test 1

Day 13 (3.6) 76

Day 14 (3.9) 82

Day 15 (3.9, 3.10) 90

Day 16 (5.1, 5.2) 96

Day 17 (4.9) 104

Day 18 (5.3) 111

Day 19 (5.5) 119

Days 20 – Project Day

Day 21 (4.1) 127

Days 22 & 23 – Review & Test 2

Day 24 (4.3) 133

Day 25 (4.7) 140

Day 26 (4.5, 4.6) 147

Days 27 & 28 – Final Review

Algebra Skills Labs 154

Name: Deewang

Day 1. Zoology of Functions

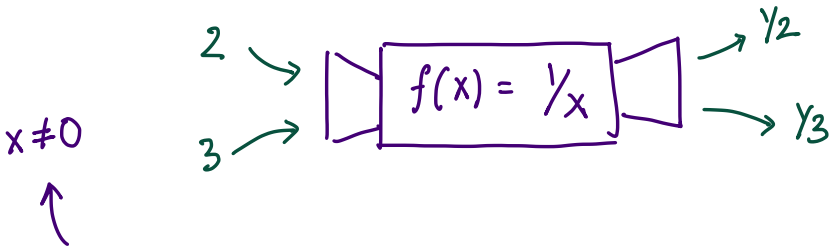
(based on Sections 1.1, 1.2, 1.3)

Learning Objectives

- Understand and recognise the four representations of a function: formula, table, graph, and description.
- Identify parent functions including Linear, Power, Polynomial, Rational, and Trigonometric.
- Apply vertical and horizontal transformations to existing functions.

What is a **function**?

a rule that assigns to each valid input, exactly one output



domain = collection of valid inputs **range** = valid outputs

Representing Functions. There are **four ways** to represent a function:

Analytically (formula)

$f(x) = x^2 + 1$

($f(x)$ is y)

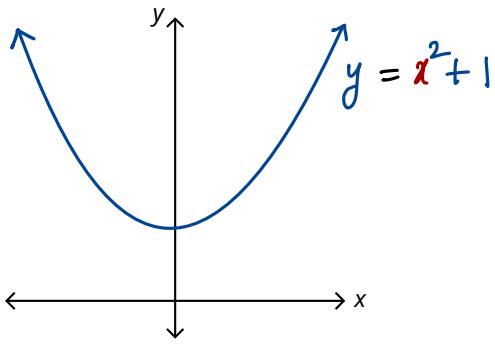
Verbally (description)

Take the input x, square it and add one to get the output.

Numerically (tabular)

x	$y = x^2 + 1$
-1	$(-1)^2 + 1 = 2$
0	$(0)^2 + 1 = 1$
1	$(1)^2 + 1 = 2$
2	$(2)^2 + 1 = 5$

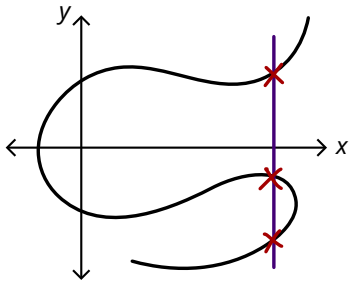
Graphically



How can we tell when something is **not a function**?

If an input has multiple outputs.

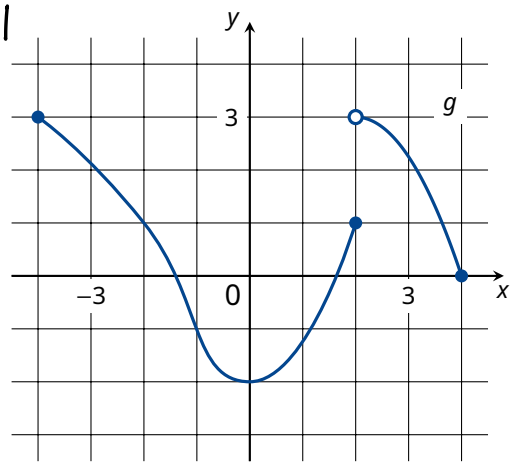
Visually, if the curve fails the vertical line test



Problem 1. Use the graph of the function g to answer the following questions.
Recall: **a hole** means a function is *not defined* at that point, while a **filled-in point** means a function *is defined* at that point.

(a) State the values of $g(-2), g(0), g(2)$ and $g(3)$.

$g(-2) = 1$, $g(0) = -2$, $g(2) = 1$, $g(3) \approx 2.1$



(b) For what x value(s) is $g(x) = 3$?

Only $x \approx 2.1$

(c) For what x value(s) is $g(x) \leqslant 2$?

$[-2.9, 2] \cup [3.2, 4]$

(d) On what interval(s) is g increasing?

$[0, 2]$

I have made some approximations!

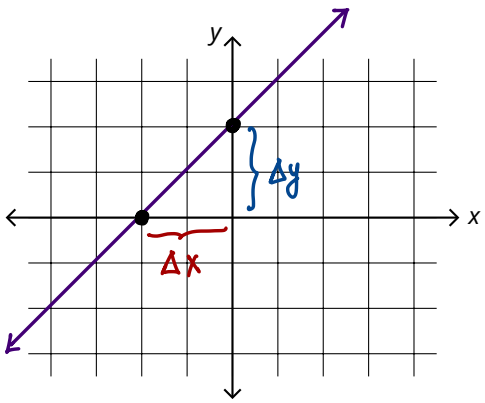
Parent Functions I.

Linear Functions I. *Slope-Intercept Form*

$y = f(x) = mx + b$ → y-intercept $= f(0)$

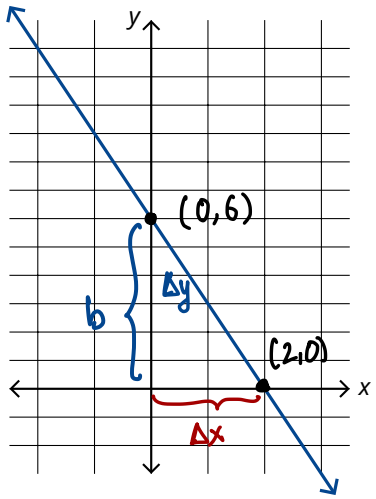
slope

$m = \Delta y / \Delta x = \dfrac{y_2 - y_1}{x_2 - x_1}$



$y = x + 2$

Example A. Find the equation of the line whose graph is given below:



$m = \Delta y / \Delta x = \frac{6-0}{0-2} = \frac{6}{-2} = -3$

$b = 6$

Equation: $y = -3x + 6$

Linear Functions II. *Point-Slope Form*

The equation of a line if you already know, or can compute, its slope m and a point (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

Example B. Find the equation of the line passing through the points $(1, 4)$ and $(3, 8)$.

$m = \Delta y / \Delta x$

$= \frac{8-4}{3-1} = \frac{4}{2} = 2$

$y - 4 = 2(x - 1)$

$y - 4 = 2x - 2$

$y = 2x + 2$ ⭐

OR

$y - 8 = 2(x - 3)$

$y - 8 = 2x - 6$

$y = 2x + 2$ ⭐

Power Functions.

$y = x^2$ domain: $(-\infty, \infty)$

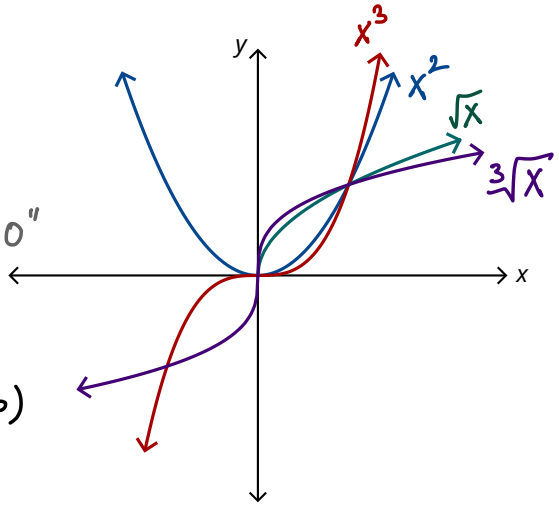
$y = x^3$ domain: $(-\infty, \infty)$

root functions

"$x > 0$"

$y = \sqrt{x} = x^{1/2}$ domain: $(0, \infty)$

$y = \sqrt[3]{x} = x^{1/3}$ domain: $(-\infty, \infty)$



Polynomial Functions.

$y = x^{2} - 5x + 6$

polynomial of deg. 2

$y = x^{3} - 4x^{2} - x + 4$

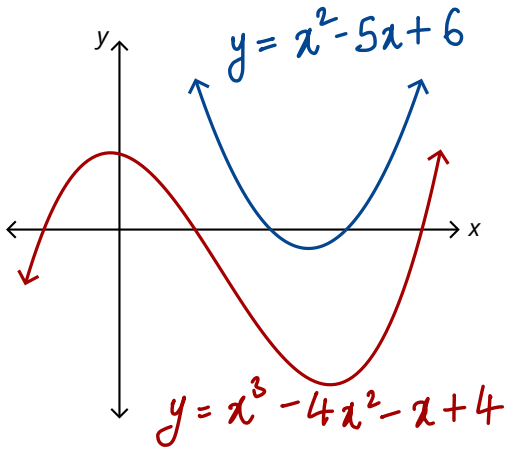
polynomial of deg 3

Linear functions

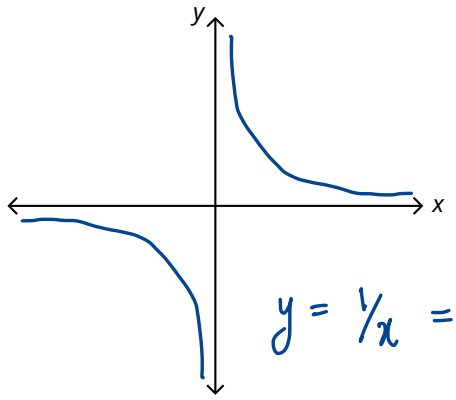
poly. of deg 1

domain :

$(-\infty, \infty)$



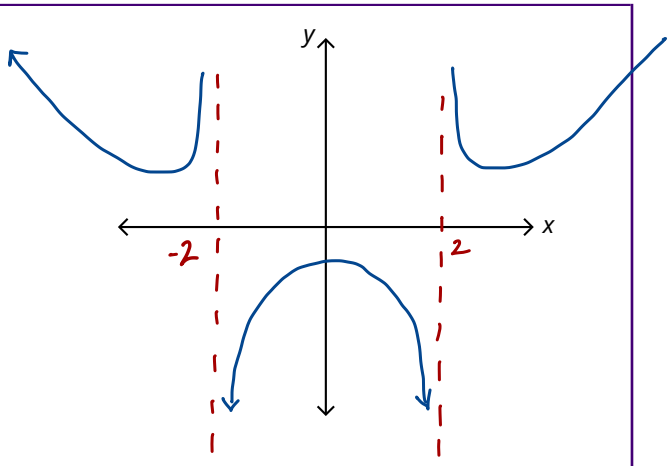
Rational Functions.



$y = 1/x = x^{-1}$

Domain : $x \neq 0$

$\rightsquigarrow (-\infty, 0) \cup (0, \infty)$



$y = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$; Domain : $x^2 - 4 \neq 0$

$\rightsquigarrow (x-2)(x+2) \neq 0 \rightsquigarrow x \neq \pm 2$

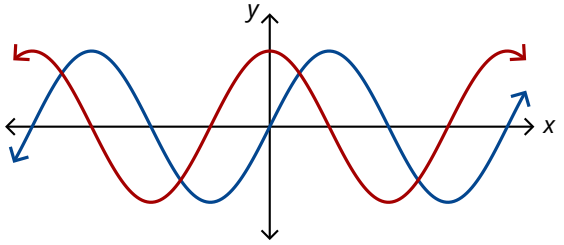
$\rightsquigarrow (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Trigonometric Functions.

	Period	Domain	Range
$\sin(x)$	2π	$(-\infty, \infty)$	$[-1, 1]$
$\cos(x)$	2π	$(-\infty, \infty)$	$[-1, 1]$
$\tan(x)$	π		$(-\infty, \infty)$

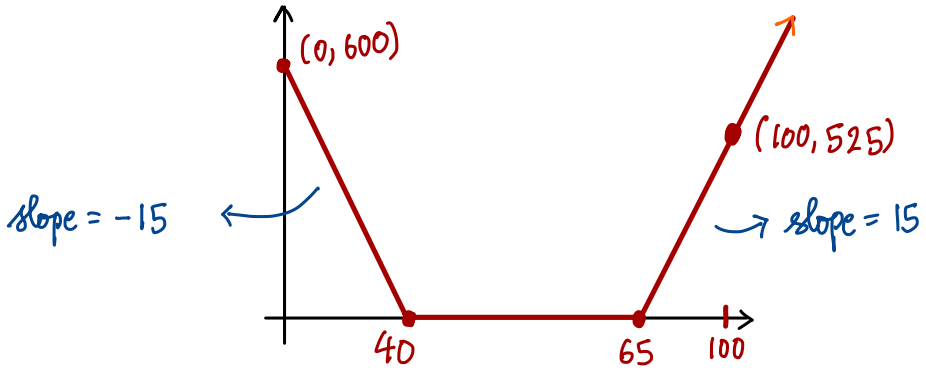
$\tan x = \sin x / \cos x$

$\cos x \neq 0$: $x \neq n\pi + \dfrac{\pi}{2}$, $n = \ldots, -2, -1, 0, 1, 2, \ldots$



Problem 2. On a certain highway, the maximum allowed speed is 65 mph, and the minimum allowed speed is 40 mph. The fine for violating these limits is \$15 for every mile per hour above the maximum speed or below the minimum speed.

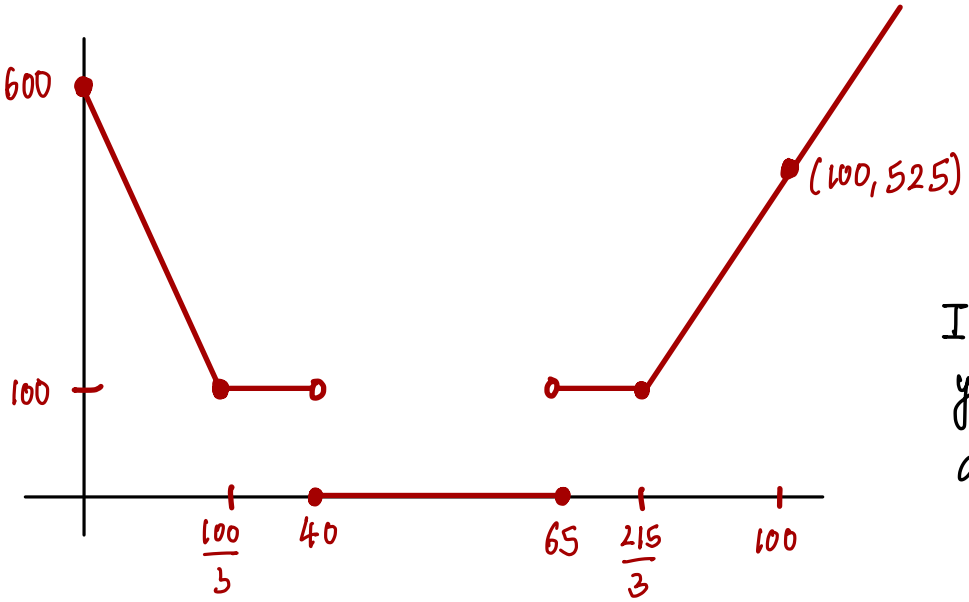
(a) Draw a graph of the fine F as a function $F(x)$ of the speed x for $0 \leqslant x \leqslant 100$.



(b) Express the fine F as a function $F(x)$ of the speed x for $0 \leqslant x \leqslant 100$.

$$F(x) = \begin{cases} -15(x-40) & 0 \leq x \leq 40 \\ 0 & 40 \leq x \leq 65 \\ 15(x-65) & 65 \leq x \leq 100 \end{cases}$$

(c) Suppose that there is a minimum fine of \$100 for any traffic violation. How would answers to the previous questions change?



I leave it to you to write this as a Piecewise Function.

Transformations of Functions

Given a function $y = f(x)$, we may **transform** the function by scaling or shifting the input or output.

Vertical: Shifts & Stretches.

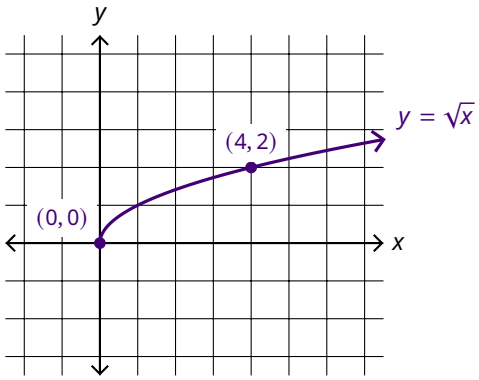
Suppose $c > 0$. To obtain the graph of

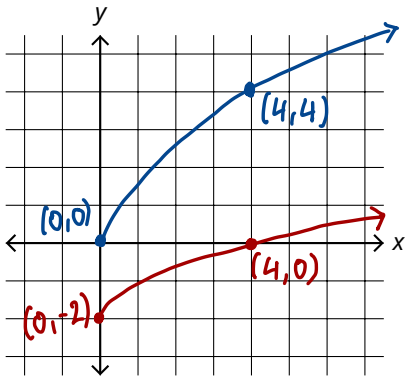
- $y = f(x) + c$, **shift** the graph of $y = f(x)$ a distance c units **upward**
- $y = f(x) - c$, **shift** the graph of $y = f(x)$ a distance c units **downward**

Suppose $c > 1$. To obtain the graph of

- $y = cf(x)$, **stretch** the graph of $y = f(x)$ vertically by a factor of c
- $y = \frac{1}{c} f(x)$, **shrink** the graph of $y = f(x)$ vertically by a factor of c

Example C. Consider the function $f(x) = \sqrt{x}$. Let us sketch $2f(x) = 2\sqrt{x}$ and $f(x) - 2 = \sqrt{x} - 2$.





Horizontal: Shifts & Stretches.

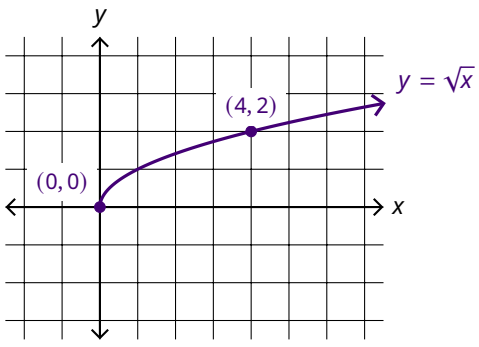
Suppose $c > 0$. To obtain the graph of

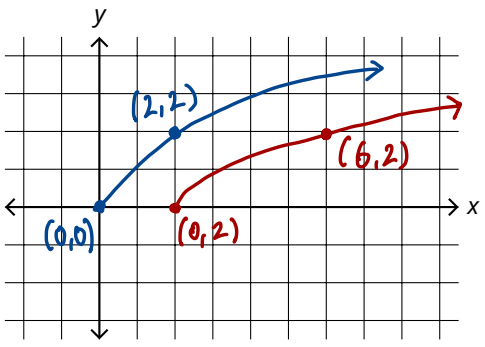
- $y = f(x - c)$, **shift** the graph of $y = f(x)$ a distance c units to the **right**
- $y = f(x + c)$, **shift** the graph of $y = f(x)$ a distance c units to the **left**

Suppose $c > 1$. To obtain the graph of

- $y = f\left(\frac{x}{c}\right)$, **stretch** the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(cx)$, **shrink** the graph of $y = f(x)$ horizontally by a factor of c

Example D. Consider the function $f(x) = \sqrt{x}$. Let us sketch $f(2x) = \sqrt{2x}$ and $f(x - 2) = \sqrt{x - 2}$.

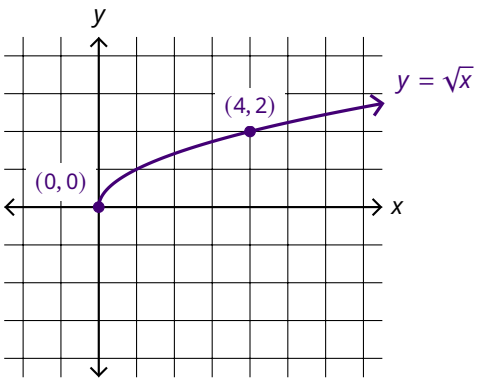


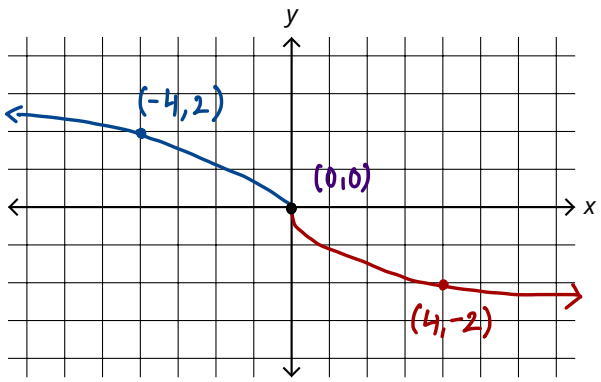


Reflections. To obtain the graph of

- $y = -f(x)$, **reflect** the graph of $y = f(x)$ **about the *x*-axis**
- $y = f(-x)$, **reflect** the graph of $y = f(x)$ **about the *y*-axis**

Example E. Consider the function $f(x) = \sqrt{x}$. Let us sketch $f(-x) = \sqrt{-x}$ and $-f(x) = -\sqrt{x}$.





Problem 3. The graph of the original function f is given in blue (and marked as f).

Match each of the following transformations of f to its graph.

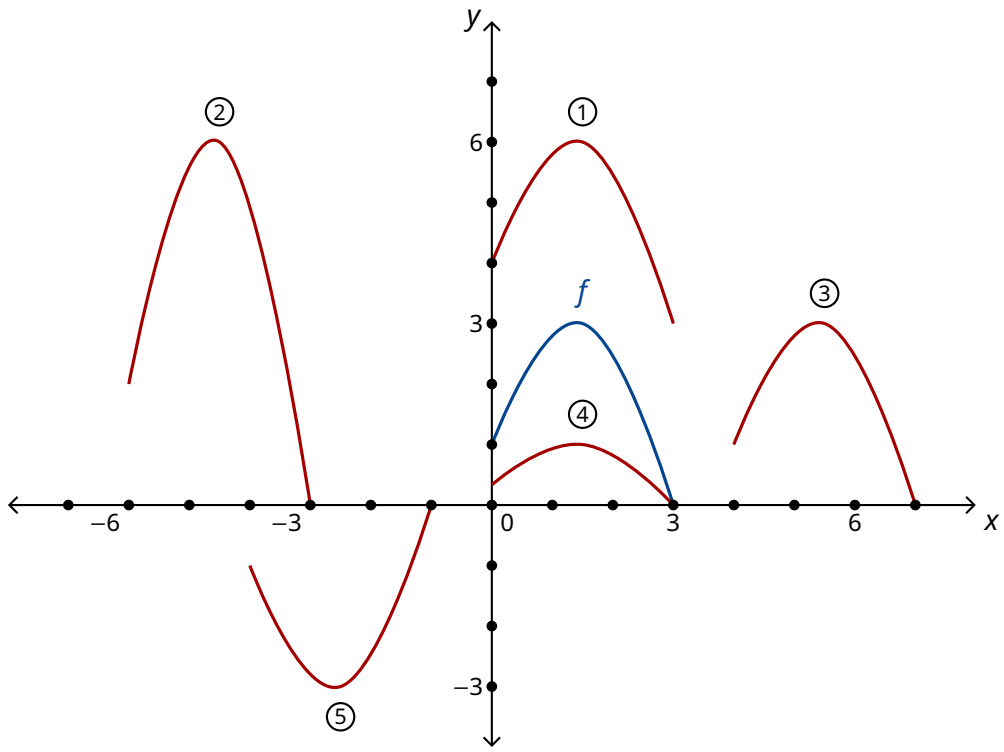
(a) $y = f(x - 4)$ 3

(b) $y = f(x) + 3$ 1

(c) $y = \frac{1}{3}f(x)$ 4

(d) $y = -f(x + 4)$ 5

(e) $y = 2f(x + 6)$ 2



Even and Odd Functions

Even and Odd Functions. Even and odd functions are functions with some nice symmetry.

- A function f is **even** if

$$f(-x) = f(x)$$

  Even functions have **symmetry** about the y-axis

- A function f is **odd** if

$$f(-x) = -f(x)$$

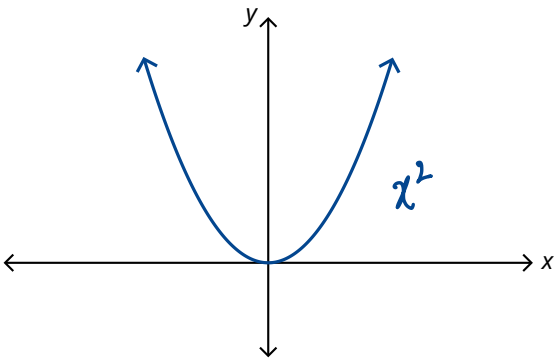
  Even functions have **symmetry** about the origin

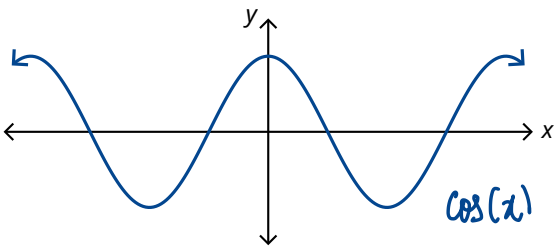
Example F.

Even Functions

$y = x^2$, $y = x^4$ } even powers

$y = \cos x$

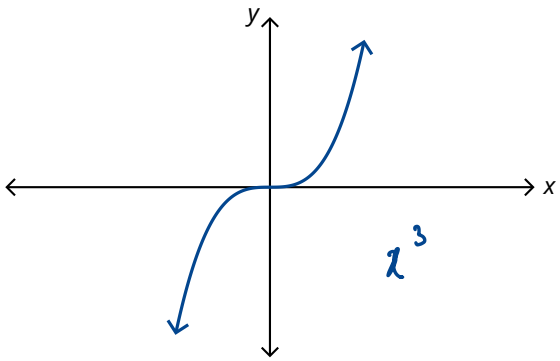


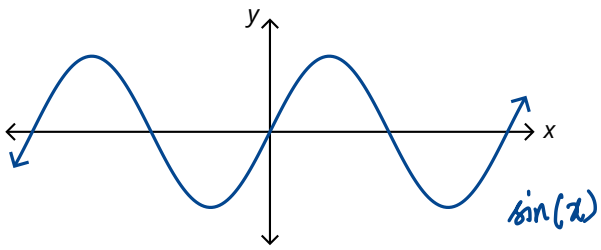


Odd Functions

$y = x^3$, $y = x^5$ } odd powers

$y = \sin x$





Name: Deewang

Day 2. Exponential, Composition& Logarithms

(based on Sections 1.3, 1.4, 1.5)

Learning Objectives

- Combine functions through composition.
- Identify and examine inverse functions.
- Master the rules of exponents and logarithms.

Parent Functions II.

Exponential Functions. For such functions the independent variable is in the exponent.

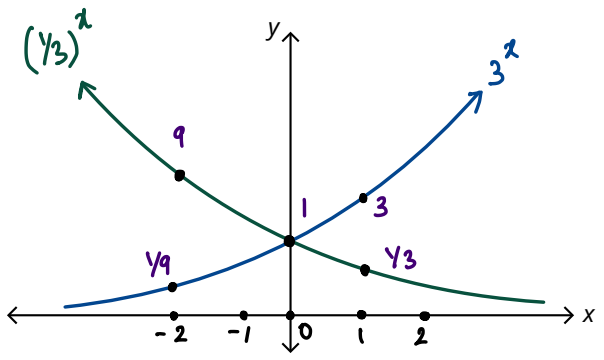
$$f(x) = a^x$$

$a > 0$ is the **base**. Recall that we always have: $a^0 = 1$

Domain of exponential functions: $(-\infty, \infty)$ **Range** of exponential functions: $(0, \infty)$

Example A.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-2	$3^{-2} = 1/3^2 = 1/9$	$(1/3)^{-2} = 3^2 = 9$
-1	$3^{-1} = 1/3$	$(1/3)^{-1} = 3$
0	$3^0 = 1$	$(1/3)^0 = 1$
1	$3^1 = 3$	$(1/3)^1 = 1/3$
2	$3^2 = 9$	$(1/3)^2 = 1/3^2 = 1/9$



Remark. If the base a is such that:

- $a > 1$, then we have exponential growth
- $0 < a < 1$, then we have exponential decay

Example B. The table below shows the estimated population over time for a bacteria culture of *Giardia lamblia*.

Time (hours)	0	4	8	12	16	20	24
Bacteria count (CFU/mL)	37	47	63	78	105	130	173

(a) Enter the data into Desmos and create a scatter plot. (**Desmos**)

(b) Find an exponential curve $f(t) = a \cdot b^t$ that fits the data well.

$$f(t) = 37 \cdot (1.06)^t$$

for what t is $f(t) = 2 \cdot 37 = 74$?

(c) Use part (b) to estimate how long it takes for the bacteria population to double. $\approx$ 11.9 hours

Exponent Rules.

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $\sqrt[x]{a} = a^{1/x}$
- $(ab)^x = a^x b^x$

Examples.

- $2^{3+1} = 2^3 \cdot 2 = 8 \cdot 2 = 16$
- $2^{-1} = \frac{1}{2}$
- $2^{4-5} = \frac{2^4}{2^5}$
- $(2^3)^4 = 2^{12}$
- $2^{2/3} = (2^2)^{1/3} = 4^{1/3} = \sqrt[3]{4}$
- $(2 \cdot 3)^4 = 2^4 \cdot 3^4$

Problem 1. Algebraic manipulation of exponents is fundamental to working with exponential functions. Use exponent rules to simplify these expressions into a single exponential expression and fill in the boxes.

(a) $\frac{(-3)^6}{9^6} = 3^{\boxed{-6}}$

$\frac{(-3)^6}{9^6} = \frac{(-1)^6 \cdot 3^6}{(3^2)^6} = 3^{6-12}$

(b) $\frac{1}{\sqrt[4]{x^5}} = x^{\boxed{-5/4}}$

$\frac{1}{\sqrt[4]{x^5}} = \frac{1}{x^{5/4}} = x^{-5/4}$

(c) $b^3(3b^{-1})^{-2} = \boxed{1/9} \cdot b^{\boxed{5}}$

$b^3 \cdot (3b^{-1})^{-2} = b^3 \cdot 3^{-2} \cdot (b^{-1})^{-2} = \frac{1}{3^2} \cdot b^3 \cdot b^2$

(d) $\frac{3a^{3/2}a^{1/2}}{a^{-1}} = \boxed{3} \cdot a^{\boxed{3}}$

$\frac{3a^{3/2} \cdot a^{1/2}}{a^{-1}} = 3 \cdot a^{3/2 + 1/2 - (-1)}$

Composition of Functions

Given two functions f and g, we may apply these rules consecutively, this is called **composition**

$$(f \circ g)(x) = f(g(x))$$

$x \to \boxed{g} \to g(x) \to \boxed{f} \to f(g(x))$

$x \to \boxed{f} \to f(x) \to \boxed{g} \to g(f(x))$

DIFFERENT!

Example C. Given $f(x) = \sqrt{x+5}$ and $g(x) = \cos(x)$. We will find

(a) $f(g(x))$

$= \sqrt{g(x)+5}$

$= \sqrt{\cos x + 5}$

(b) $(g \circ f)(x)$

$= g(f(x))$

$= \cos(f(x))$

$= \cos(\sqrt{x+5})$

(c) $f(f(x))$

$= \sqrt{f(x)+5}$

$= \sqrt{\sqrt{x+5}+5}$

Problem 2. Let $f(x) = \sqrt{x+5}$, $g(x) = \cos(x)$ and $h(x) = x^2$. Find

(d) $f(h(x))$

$= \sqrt{h(x)+5} \quad = \sqrt{x^2+5}$

(e) $(h \circ g)(x) \quad = h(g(x))$

$= (g(x))^2 = (\cos(x))^2 = \cos^2(x)$

(f) $h(f(g(x)))$

$= (f(g(x)))^2 = (\sqrt{g(x)+5})^2 = g(x)+5 = \cos(x)+5$

Example D. We will *decompose* $F(x) = \sqrt{(\sin(x)+3)^4 + 2}$

- Most outside : $f(x) = \sqrt{x}$ peel the layer $\rightsquigarrow$ $(\sin(x)+3)^4 + 2$ remains
- Most outside : $g(x) = x+2$ peel the layer $\rightsquigarrow$ $(\sin(x)+3)^4$ remains
- Most outside : $h(x) = x^4$ peel the layer $\rightsquigarrow$ $\sin(x)+3$ remains
- $k(x) = \sin(x)+3$ $\rightsquigarrow$ $F(x) = f(g(h(k(x))))$

Problem 3. Use the graphs of f and g to evaluate each expression, or explain why it is undefined.

Tip. First find the *inside*, e.g., for (a), first figure out what $g(2)$ is.

(a) $f(g(2))$

$= f(5)$

$= 4$

(b) $g(f(0))$

$= g(0) = 3$

(c) $(f \circ g)(0)$

$= f(3) = 0$

(d) $(g \circ f)(6)$

$= g(6)$

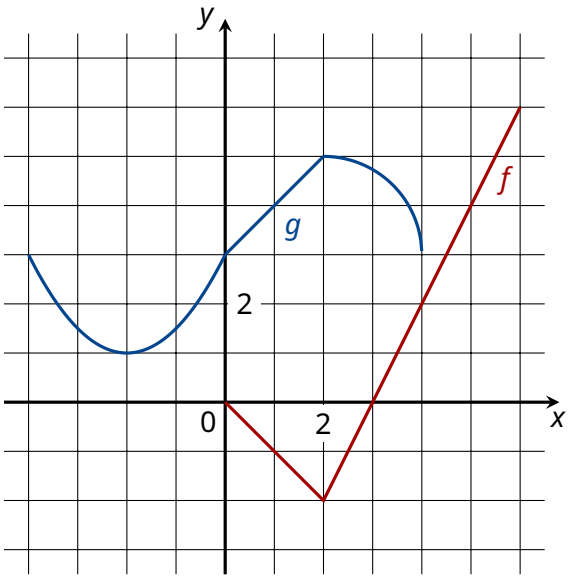
$=$ undefined

(e) $(g \circ g)(-2)$

$= g(1) = 4$

(f) $(f \circ f)(4)$

$= f(2) = -2$



Inverse Functions

Inverse Functions. For a function f, its **inverse function** f^{-1} is a rule that undoes f and vice versa.

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

For example, the inverse function of $f(x) = x + 1$ is

$f^{-1}(x) = x - 1$ ⇝

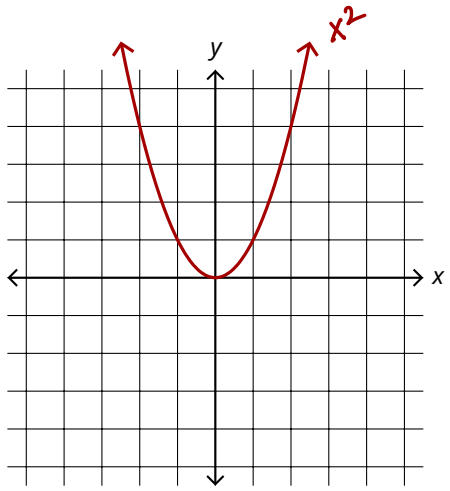
$f^{-1}(f(x)) = f(x) - 1 = (x+1) - 1 = x$

$f(f^{-1}(x)) = f^{-1}(x) + 1 = (x-1) + 1 = x$

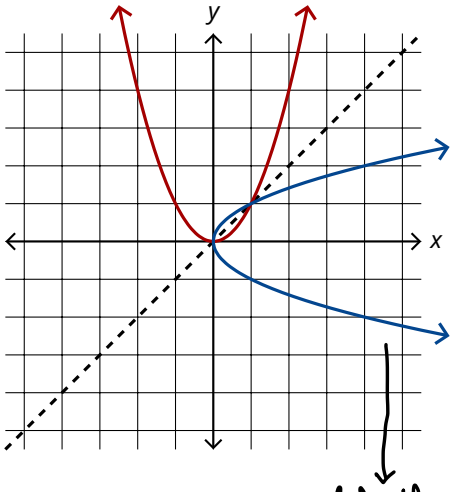
Main Properties.

- Domain of f^{-1} = range of f.
- Range of f^{-1} = domain of f.
- The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

Example E. Consider the function $f(x) = x^2$

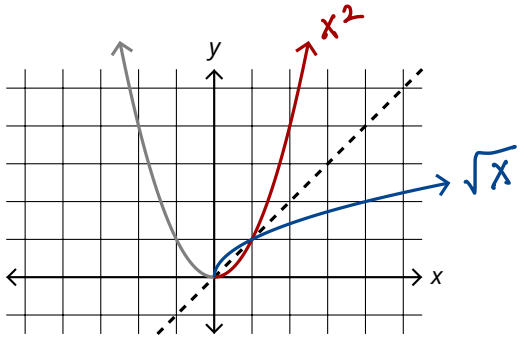


reflecting along the y = x line ⇝



fails the vertical line test

Functions that pass the horizontal line test have inverses!



Parent Functions III.

Logarithmic Functions. The graph of an exponential function a^x *always passes* the **horizontal line test**.

Its *inverse function* is the **logarithm function** with base $a > 0$

$$\log_a(x)$$

Domain of logarithmic functions: $(0, \infty)$ **Range** of logarithmic functions: $(-\infty, \infty)$

Main Property: exponential and logarithmic functions *undo* each other

$$\log_a(a^x) = x \qquad a^{\log_a(x)} = x$$

In particular, we may use logarithms to *solve exponential equations*:

$a^y = x$ ⇝ apply $\log_a$ ⇝ $\log_a(a^y) = \log_a(x)$ ⇝ $y = \log_a(x)$

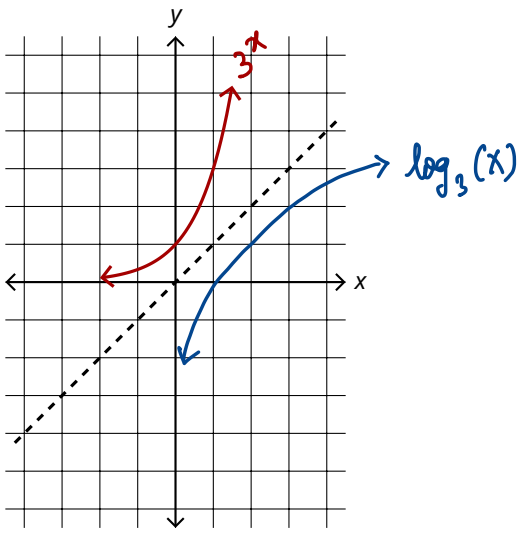
Therefore,

$a^0 = 1$ ⇝ $\log_a(1) = 0$

Example F. Graph of logarithmic functions:

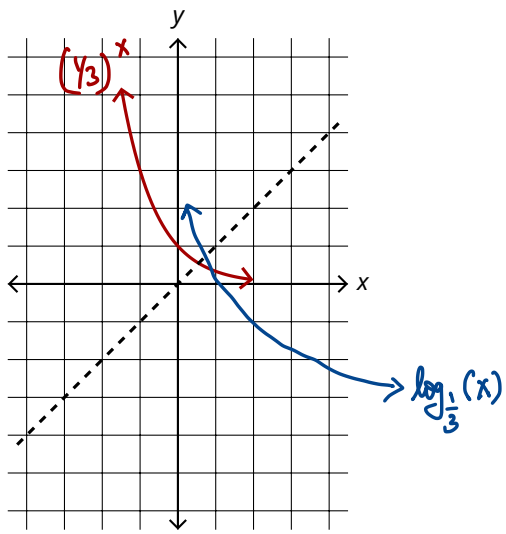
$f(x) = \log_3(x)$

x	$y = \log_3(x)$
3^{-2}	$\log_3(3^{-2}) = -2$
3^{-1}	$\log_3(3^{-1}) = -1$
$1 = 3^0$	$\log_3(1) = 0$
3^1	$\log_3(3^1) = 1$
3^2	$\log_3(3^2) = 2$



$f(x) = \log_{1/3}(x)$

x	$y = \log_{1/3}(x)$
$(1/3)^{-2}$	$\log_{1/3}((1/3)^{-2}) = -2$
$(1/3)^{-1}$	$\log_{1/3}((1/3)^{-1}) = -1$
$1 = (1/3)^0$	$\log_{1/3}(1) = 0$
$(1/3)^1$	$\log_{1/3}((1/3)^1) = 1$
$(1/3)^2$	$\log_{1/3}((1/3)^2) = 2$



Logarithm Rules.

- $\log_a(a^x) = x$
- $a^{\log_a(x)} = x$
- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(x/y) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r \cdot \log_a(x)$

Examples.

- $10^{\log_{10}(3)} = 3$
- $\log_{10}(20) + \log_{10}(5) = \log_{10}(20 \cdot 5) = \log_{10}(100) = \log_{10}(10^2) = 2$
- $\log_{10}(500) - \log_{10}(5) = \log_{10}(500/5) = \log_{10}(100) = 2$
- $\log_{10}(5^7) = 7 \cdot \log_{10}(5)$

Problem 4. Find the exact value of each expression below, referring to the table above as needed to do algebraic manipulations. **(No calculator necessary.)**

(a) $\log_3\left(\frac{1}{81}\right)$ (write $1/81$ as $3^{\square}$)

$= \log_3(3^{-4}) = -4$

(b) $\log_5 \sqrt{5}$

$\log_5(5^{1/2}) = \frac{1}{2}$

(c) $\log_3 10 - \log_3 5 - \log_3 18$

$\log_3\left(\frac{10}{5\cdot 18}\right) = \log_3\left(\frac{1}{9}\right) = \log_3(3^{-2})$

$= -2$

(d) $2^{-\log_2(5)}$ (write $-\log_2(5)$ as $\log_2 \square$)

$2^{\log_2(5^{-1})} = 5^{-1} = \frac{1}{5}$

Example G. Let us revisit (c) from Example B using the exponential function we discovered in (b).

$f(t) = 37\cdot(1.06)^t$. (c) $\rightsquigarrow$ Solve $f(t) = 74$

$37\cdot(1.06)^t = 74 \rightsquigarrow (1.06)^t = 2$

$t = \log_{1.06}(2) \approx 11.9$

The Exponential and Logarithmic Functions.

e is **Euler's number** and $e \approx 2.71828$. The inverse of e^x is the **natural logarithm**, denoted

Problem 5. Solve for x

(a) $e^{2x+3} - 7 = 0$

$e^{2x+3} = 7$

$2x+3 = \ln(7)$

$x = \frac{\ln(7)-3}{2}$

(b) $\ln(5-2x) = -3$

$\ln(5-2x) = -3$

$5-2x = e^{-3}$

$x = \frac{5-e^{-3}}{2}$

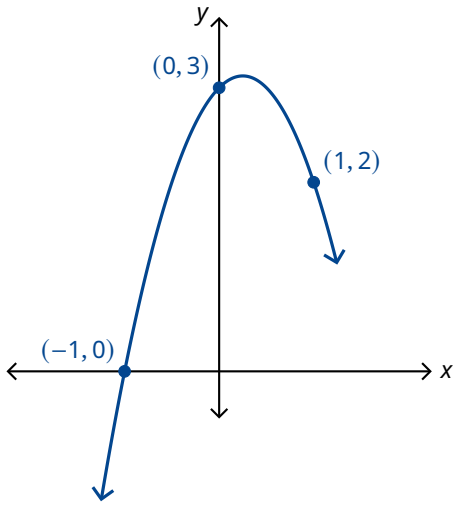
Recap Problems

The aim is to review Day 1 material; so have the filled-in Day 1 notes in front of you as a reference!

Recap Problem 1. What is the domain of the function $\ln(x+6)$?

Note. The domain of ln is the set of all positive numbers, i.e., you are only allowed to taken ln of a positive number.

Recap Problem 2. A quadratic polynomial is of the form $f(x) = ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$. Given below is the graph of $f(x)$, use it to determine $f(x)$, i.e., determine a, b, c.



Name: Deewang

Day 3. The Tangent & Velocity Problem and Limits

(based on Sections 2.1, 2.2)

Learning Objectives

- Understand how limits arise naturally from the tangent and velocity problems.
- Distinguish between one-sided limits, two-sided limits, and limits that do not exist.
- Evaluate limits using numerical tables, graphical information, and algebraic simplification.

The Story of Two Problem

The Velocity Problem. Suppose you were concerned about the gas mileage of your vehicle and checked your odometer every hour, and realised that the distance s, in miles, travelled each hour is given by the function

$$s(t) = 30\sqrt{t}$$

- What was your average velocity over $3 \leqslant t \leqslant 4$ hours?

$$\text{Avg. velocity} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(4) - s(3)}{4-3} = \frac{30\sqrt{4} - 30\sqrt{3}}{1}$$

$$= 30(2-\sqrt{3}) \approx 8.04$$

- What was your average velocity over $4 \leqslant t \leqslant 5$ hours?

$$\text{Avg. velocity} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(5) - s(4)}{5-4} = \frac{30\sqrt{5} - 30\sqrt{4}}{1}$$

$$= 30(\sqrt{5}-2) \approx 7.08$$

- Estimate the instantaneous velocity at time $t = 4$ hours: what is the velocity of your vehicle at **exactly** 4 hours?

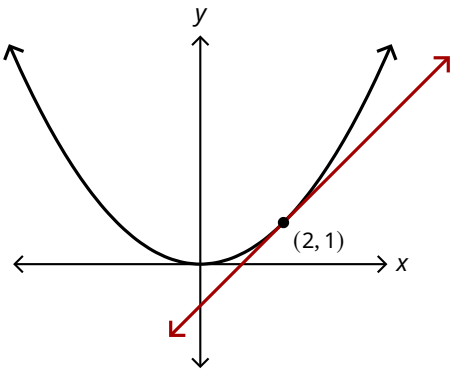
x	*avg. velocity over x and 4*	x	*avg. velocity over x and 4*
5	7.08	3	8.04
4.1	$\frac{s(4.1)-60}{4.1-4} \approx 7.45$	3.9	$\frac{s(3.9)-60}{-0.1} \approx 7.55$
4.01	$\frac{s(4.01)-60}{4.01-4} \approx 7.49$	3.99	$\frac{s(3.99)-60}{0.01} \approx 7.5$

Numerical evidence suggests (e.g., take the average) the instantaneous velocity is $\boxed{7.5}$

Let $s(t)$ be the position of an object at time t. Then, the **instantaneous velocity** of the object at time $t = a$ is

$$\lim_{x \to a} \underbrace{\frac{s(x) - s(a)}{x - a}}_{\text{difference quotient}}$$

The Tangent Problem. We wish to find the equation of the **tangent line** to $f(x) = \frac{x^2}{4}$ at the point $(2, 1)$.



What do we need to find the equation of the line?

a point on the line: (1, 1) & slope ???

We approx. by computing the slope of the line between $(2, 1)$ and points close to it

slope of the line from $(2, 1) = (2, f(2))$ *to* $(2 + h, f(2 + h))$:

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - 1}{h}$$

$h \geqslant 0$	$2 + h$	*slope of the line from* $(2, 1)$ *to* $(2 + h, f(2 + h))$	$h \leqslant 0$	$2 + h$	*slope of the line from* $(2, 1)$ *to* $(2 + h, f(2 + h))$
0.1	2.1	$\frac{f(2.1)-1}{0.1} = 1.025$	-0.1	1.9	$\frac{f(1.9)-1}{-0.1} = 0.975$
0.01	2.01	$\frac{f(2.01)-1}{0.01} = 1.0025$	-0.01	1.99	$\frac{f(1.99)-1}{-0.01} = 0.9975$

Numerical evidence suggests (e.g., take the average) that the slope is $\boxed{1}$.

Equation of the tangent line: $y - 1 = 1(x - 2) \rightsquigarrow y = x - 1$

For a function $f(x)$ and a point a in its domain. The **slope of the tangent line** to $y = f(x)$ at $(a, f(a))$ is

$$= \lim_{h \to 0} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\text{difference quotient}}$$

Problem 1. Here is a table giving the position of a motorcyclist after accelerating from rest.

t (seconds)	1	2	3	4	5	6
s (feet)	4.9	20.6	46.5	79.2	124.8	176.7

(a) Find the average velocity of the motorcyclist over the time periods $[2,4]$, $[2,3]$, and $[3,4]$.

On $[2,4]$

$$\frac{s(4)-s(2)}{4-2} = \frac{79.2-20.6}{2} = 29.3 \text{ ft/s}$$

On $[2,3]$

$$\frac{s(3)-s(2)}{3-2} = \frac{46.5-20.6}{1} = 25.9 \text{ ft/s}$$

On $[3,4]$

$$\frac{s(4)-s(3)}{4-3} = \frac{79.2-46.5}{1} = 32.7 \text{ ft/s}$$

(b) Estimate the instantaneous velocity of the motorcyclist at $t = 3$ using the average velocities you computed in (a).

We'll average the velocities over $[2,3]$ and $[3,4]$

$(25.9 + 32.7)/2 = 29.3$ ft/s This is one approximation

(c) Suppose that you are told that the actual position of the motorcyclist is given by the function

$$s(t) = 30t - 24.$$

Use this function to again estimate the instantaneous velocity of the motorcyclist at $t = 3$. Do you think this estimate is better than the one you made in (b)?

We will find the average velocity over smaller intervals

- $[2.999, 3]$ $\quad \frac{s(3)-s(2.999)}{2.999-3} = 29.585$
- $[3, 3.001]$ $\quad \frac{s(3.001)-s(3)}{3.001-3} = 29.594$

averaging ≈ 29.59 ft/s which is expectdly a better estimate

Limits

Limits. Let $f(x)$ be a function and a a real number not necessarily in the domain of f. We write

$$\lim_{x \to a} f(x) = L$$

and read it as "*L is the* **limit** *of $f(x)$ as x approaches a*", if for inputs x close to a, the output $f(x)$ gets close to L.

We do not care what is happening at $x = a$ itself.

If we are solely looking at inputs x which are close to a but to the **left** of a, and the output gets close to L, we have the **left-hand limit**

$$\lim_{x \to a^-} f(x) = L$$

If we are solely looking at inputs x which are close to a but to the **right** of a, and the output gets close to L, we have the **right-hand limit**

$$\lim_{x \to a^+} f(x) = L$$

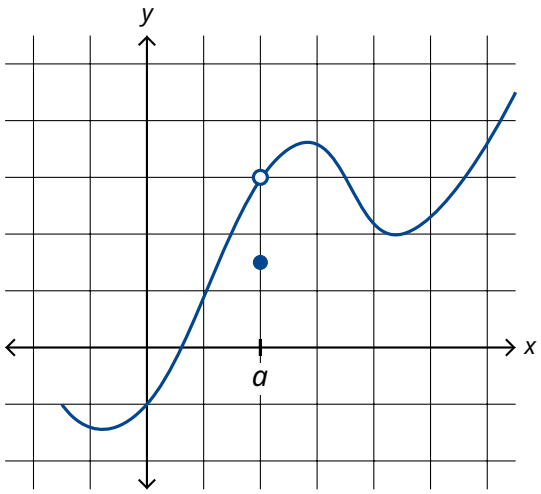
Fact. The limit $\lim_{x\to a} f(x)$ exists exactly when both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x)$$

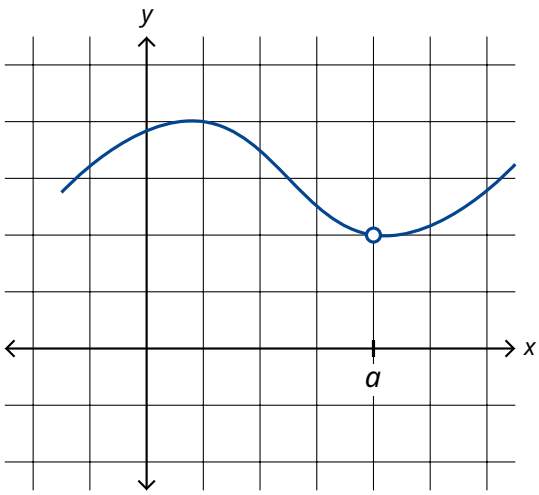
Therefore,

if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ **does not exist**.

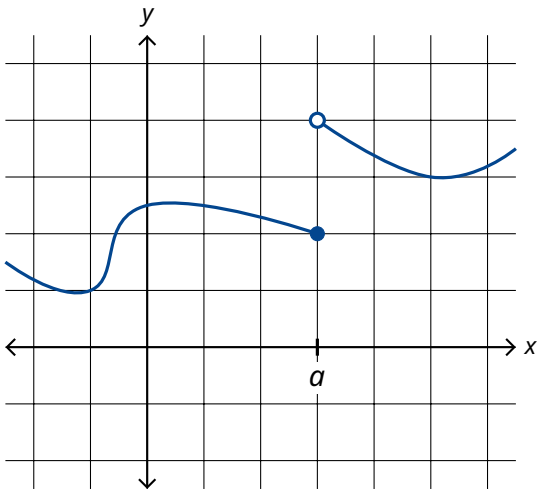
Example A.



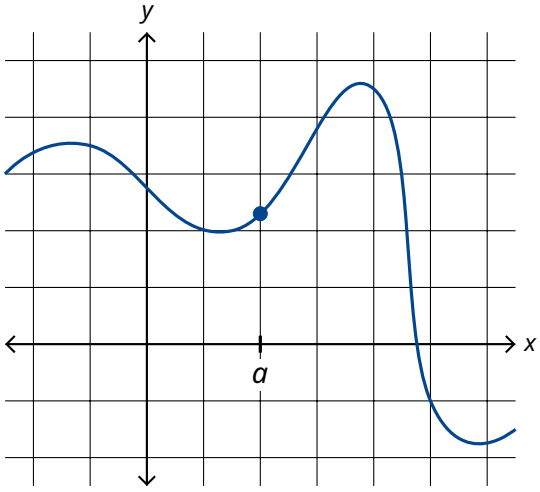
$f(a) \neq \lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$



$f(a) \quad \lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$



$f(a) \quad \lim_{x\to a} f(x) \quad \lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$



$f(a) = \lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

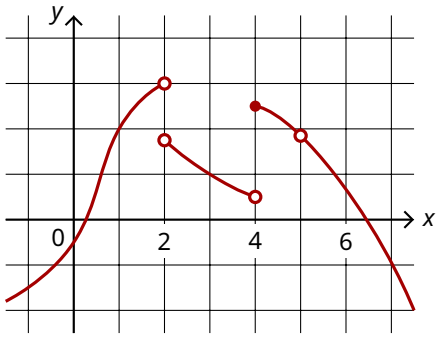
Problem 2. For the function g shown below, find a number a that satisfies each of the descriptions:

(a) $\lim_{x\to a} g(x)$ exists but $g(a)$ is not defined. $a = 5$

(b) $\lim_{x\to a} g(x)$ does not exist but $g(a)$ is defined. $a = 4$

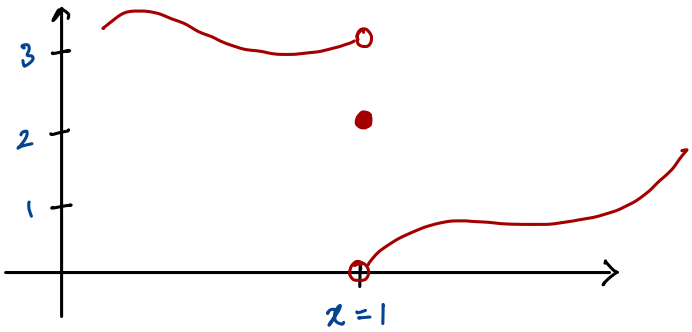
(c) $\lim_{x\to a^-} g(x)$ and $\lim_{x\to a^+} g(x)$ exist but $\lim_{x\to a} g(x)$ does not exist. $a = 2$

(d) $\lim_{x\to a^+} g(x) = g(a)$ but $\lim_{x\to a^-} g(x) \neq g(a)$. $a = 4$



Problem 3. Sketch the graph of a function f that satisfies the conditions

$$\lim_{x\to 1^-} f(x) = 3, \qquad \lim_{x\to 1^+} f(x) = 0, \qquad \text{and} \qquad f(1) = 2.$$



Problem 4. Find each of the limits below (or find that the limit does not exist) by making a table of values of the function as x approaches the value where you want the limit using Desmos.

For example, you can do the following for (a), and repeat these steps on Desmos for (b) and (c):

- On Desmos, write $f(x) = \dfrac{\ln x - \ln 4}{x - 4}$
- Go to the next cell, and click the plus sign on the top left and select table from the dropdown menu.
- Change column titles from x_1 to x, and from y_1 to $f(x)$.
- Populate the first column with values close to 4 and the second column will automatically give you function outputs.
- Looking at the outputs, make an approximation as to what the limit could be, or if the limit does not exist, and record it here.

(a) $\lim_{x\to 4} \dfrac{\ln x - \ln 4}{x - 4}$

$= 0.25$

Numerical evidence suggest 0.25

(b) $\lim_{x\to 0} x\sqrt{1 + x^{-2}}$

does not exist

Numerical evidence suggests left-side and right-side limits do not agree.

(c) $\lim_{x\to 0^+} \ln(\sin x)$

$= -\infty$

Numerical evidence suggests that the value is getting more and more negative as $x \to 0^+$.

Computing Limits

Fact. If $f(x)$ is a parent function and a is in the domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$.

Example B. (a) $\lim_{x\to 5} x^2 - 9 = 5^2 - 9 = 16$ (b) $\lim_{x\to 3} \frac{x-3}{x+3} = \frac{3-3}{3+3} = 0$

Warning. The fact above is **not true** if a under the limit in $x \to a$ is *not* the domain of $f(x)$.

If your substitution of a will result in an illegal move – e.g., **zero in the denominator** – you should try and *algebraically simplify first* to get rid off what makes making a move illegal and then substitute a into the simplified expression.

This works since the limit is interested in what is happening *near* a and not what is happening at a.

Example C. Let us compute the following limits.

(a) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

Cannot plug in $x=5$!!! $\rightsquigarrow \frac{5-5}{5^2-25} = \frac{0}{0}$ (Big NO!)

$$= \lim_{x\to 5} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5} \frac{1}{x+5} \overset{\text{plug in}}{=} \frac{1}{5+5} = \frac{1}{10}$$

$a^2 - b^2 = (a-b)(a+b)$

Problem 5. Compute the limit:

$$\lim_{x\to 1} \frac{x^2+4x-5}{x^2-4x+3}$$

Note. Can you just plug in $x = 1$? If not, trying factoring both quadratics and see if you can simplify the fraction.

$$\lim_{x\to 1} \frac{x^2+4x-5}{x^2-4x+3} = \lim_{x\to 1} \frac{(x+5)(x-1)}{(x-3)(x-1)}$$

$$= \lim_{x\to 1} \frac{x+5}{x-3}$$

$$= \frac{1+5}{1-3} = -3$$

Recap Problems

The aim is to review Day 2 material; so have the filled-in Day 2 notes in front of you as a reference!

Recap Problem 1. Decompose the function

$$F(x) = \ln(x^2 - 9)$$

as a composition of *three* functions $f(g(h(x)))$

Recap Problem 2. Solve for x in the following equation (the answer is a whole number):

$$\log_2(x) = \log_2(5 + x) - \log_2(2)$$

Note. Proceed as follows: (1) Combine the right hand-side using log rules; (2) you may neutralise/undo the logarithm using that fact that $2^{\log_2(y)} = y$; (3) now solve for x.

Name: Deewang

Day 4. Computing Limits & Continuity

(based on Sections 2.3, 2.5)

Learning Objectives

- Algebraically manipulate and apply limit laws to compute limits.
- Use the Squeeze Theorem to evaluate limits of "trapped" functions.
- Understand the conceptual ideas of continuity.

Computing Limits

Fact. If $f(x)$ is a parent/nice function and a is in the domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$.

Example A. (a) $\lim_{x\to 5} x^2 - 9$ $= 5^2 - 9 = 16$ (b) $\lim_{x\to 3} \frac{x-3}{x+3}$ $= \frac{3-3}{3+3} = 0$

Warning. The fact above is **not true** if a under the limit in $x \to a$ is *not* the domain of $f(x)$.

If your substitution of a will result in an illegal move – e.g., **zero in the denominator** – you should try and *algebraically simplify first* to get rid off what makes making a move illegal and then substitute a into the simplified expression.

This works since the limit is interested in what is happening *near* a and not what is happening at a.

Example B. Let us compute the following limits.

Cannot plug in $x=5$!!! $\rightsquigarrow \frac{5-5}{5^2-25} = \frac{0}{0}$ (Big NO!)

(a) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

$= \lim_{x\to 5} \frac{x-5}{(x-5)(x+5)} = \lim_{x\to 5} \frac{1}{x+5} \overset{\text{(*)}}{=} \frac{1}{5+5} = \frac{1}{10}$

$a^2 - b^2 = (a-b)(a+b)$

Problem 1. Compute the limit:

$$\lim_{x\to 1} \frac{x^2+4x-5}{x^2-4x+3}$$

Note. Can you just plug in $x = 1$? If not, trying factoring both quadratics and see if you can simplify the fraction.

$\lim_{x\to 1} \frac{x^2+4x-5}{x^2-4x+3} = \lim_{x\to 1} \frac{(x+5)(x-1)}{(x-3)(x-1)}$

$= \lim_{x\to 1} \frac{x+5}{x-3} = \frac{1+5}{1-3} = -3$

More on Limits

Limit Laws. Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

1. $\lim_{x\to a} (c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)$
2. $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$ where $n > 0$ is an integer
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where $n > 0$ is an integer

Example C. Let us compute the following limits.

(a) $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h} \cdot \dfrac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{h\to 0} \dfrac{(\sqrt{1+h})^2 - 1^2}{h(\sqrt{1+h}+1)}$

$\not\times \dfrac{\sqrt{1+0}-1}{0}$

$= \dfrac{0}{0}$ ×

$= \lim_{h\to 0} \dfrac{(1+h)-1}{h(1+\sqrt{1+h})} = \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}(1+\sqrt{1+h})}$

$= \lim_{h\to 0} \dfrac{1}{1+\sqrt{1+h}} = \dfrac{1}{1+\sqrt{1+0}} = \dfrac{1}{2}$

(b) $\lim_{t\to -1} \sqrt{5-t^2} = \sqrt{5-(-1)^2} = \sqrt{5-1} = 2$

Problem 2. Compute the following limits.

(a) $\lim_{u\to -1^+} 1 + \sqrt{3+u^3+2u^2}$

$= 1 + \sqrt{\lim_{u\to -1^+} (3+u^3+2u^2)}$

$= 1 + \sqrt{3+(-1)^3+2(-1)^2}$

$= 1+\sqrt{4} = 3$

(b) $\lim_{t\to 1^-} \left(\dfrac{2t^5-t^4}{5t^2+4}\right)^3 = \left(\lim_{t\to 1^-} \dfrac{2t^5-t^4}{5t^2+4}\right)^3$

$= \left(\dfrac{2\cdot 1^5 - 1^4}{5\cdot 1^2+4}\right)^3$

$= \left(\dfrac{1}{9}\right)^3 = \dfrac{1}{729}$

(c) $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$ $= \lim_{h\to 0} \dfrac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \dfrac{(\sqrt{9+h})^2 - 3^2}{h(\sqrt{9+h}+3)}$

$= \lim_{h\to 0} \dfrac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)} = \lim_{h\to 0} \dfrac{1}{\sqrt{9+h}+3}$

$= \dfrac{1}{\sqrt{9+0}+3} = \dfrac{1}{6}$

(d) $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$ $= \lim_{x\to 3} \dfrac{1}{x-3}\cdot\left(\dfrac{1}{x}-\dfrac{1}{3}\right)$

$= \lim_{x\to 3} \dfrac{1}{x-3}\cdot\dfrac{3-x}{3x} = \lim_{x\to 3} \dfrac{3-x}{x-3}\cdot\dfrac{1}{3x} = \lim_{x\to 3} -\dfrac{\cancel{(x-3)}}{\cancel{x-3}}\cdot\dfrac{1}{3x}$

$= \lim_{x\to 3} -1\cdot\dfrac{1}{3x} = -1\cdot\dfrac{1}{3\cdot 3} = -\dfrac{1}{9}$

(e) $\lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \dfrac{2xh+h^2}{h}$

$= \lim_{h\to 0} \dfrac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} 2x+h = 2x$

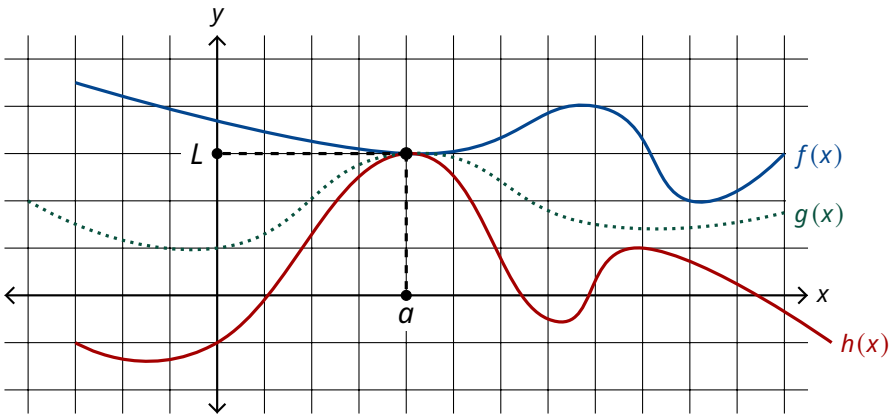
The Squeeze Theorem

The Squeeze Theorem. If $f(x) \leqslant g(x) \leqslant h(x)$ when is x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$



Example D. Let us compute $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right)$. See graph: https://www.desmos.com/calculator/xe7zcnvdfu.

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$. So, $-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\cdot\sin\left(\frac{\pi}{x}\right) \le \sqrt{x^3+x^2}$

Moreover $\lim_{x\to 0} \sqrt{x^3+x^2} = \sqrt{\lim_{x\to 0} x^3+x^2} = \sqrt{0^3+0^2} = 0$.

Similarly, $\lim_{x\to 0} -\sqrt{x^3+x^2} = 0$. So, by Squeeze $\lim_{x\to 0} \sqrt{x^3+x^2}\cdot\sin\left(\frac{\pi}{x}\right) = 0$

Problem 3. Consider the limit

$$\lim_{x\to 0} x^2\cos(2\pi x)$$

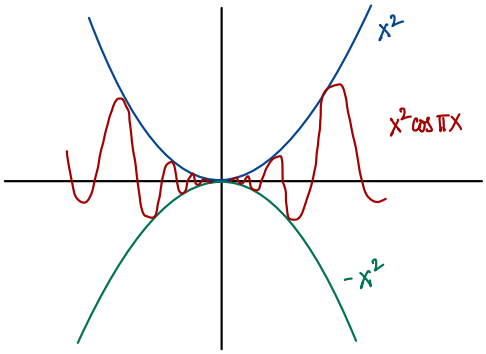
(a) What does the squeeze theorem say about this limit? Recall that $-1 \leqslant \cos(x) \leqslant 1$ for any real number x.

$-1 \le \cos 2\pi x \le 1$. So, $-x^2 \le x^2\cos 2\pi x \le x^2$

$\lim_{x\to 0} x^2 = 0^2 = 0$. Similarly, $\lim_{x\to 0}(-x^2) = 0$.

By Squeeze theorem, $\lim_{x\to 0} x^2\cos 2\pi x = 0$

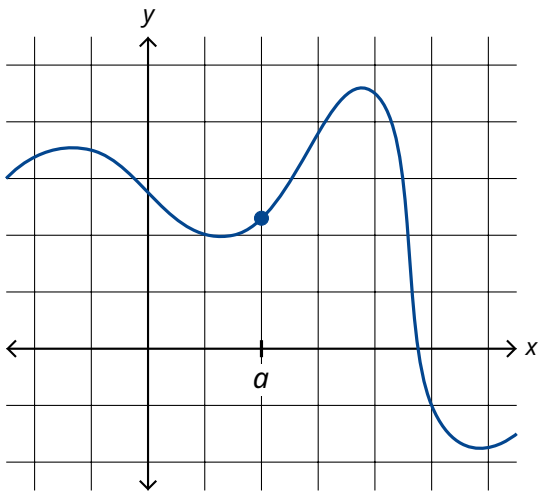
(b) Plot the function $f(x) = -x^2$, $g(x) = x^2\cos(2\pi x)$ and $h(x) = x^2$ on the same screen and roughly sketch it below.



Continuity

Continuity at a Point. Consider a function $f(x)$, and a a number *in its domain*. We say f is **continuous** at a if

$$f(a) = \lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$



Fact. *All our parent functions are continuous.*

Example E. Consider the following function

$$f(x) = \begin{cases} x^2 + a & x < 3 \\ b & x = 3 \\ 2a + x & x > 3 \end{cases}$$

Find a and b such that $f(x)$ is continuous at 3. (**playing in Desmos**)

$f(3) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

b

$3^2 + a = 2a + 3$

$9 + a = 2a + 3$

$a = 6$

And $b = 3^2 + a = 9 + 6 = 15$

Problem 4. Consider the following function

$$f(x) = \begin{cases} 2c + x + e^x & x < 2 \\ e^2 + 3c\sqrt{2x} & x \geqslant 2 \end{cases}$$

Find c such that $f(x)$ is continuous at 2.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$2c + 2 + e^2 = e^2 + 3c\sqrt{2 \cdot 2}$

$2c + 2 = 3c \cdot \sqrt{4} = 6c$

$4c = 2 \rightsquigarrow \boxed{c = 1/2}$

Problem 5. Suppose that we know

$$\lim_{x \to 1} \frac{f(x) - 8}{x - 1} = 10$$

Find $\lim_{x \to 1} f(x)$.

Let $L = \lim_{x \to 1} f(x)$

$L - 8 = \lim_{x \to 1} f(x) - 8 = \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \cdot x - 1 = \left(\lim_{x \to 1} \frac{f(x) - 8}{x - 1}\right) \cdot \left(\lim_{x \to 1} x - 1\right)$

$= 10 \cdot (1 - 1) = 0$

So, $\lim_{x \to 1} f(x) = L = 8$

Recap Problems

Recap Problem 1. Find the limit

$$\lim_{x\to 0} \frac{e^x - 1}{x}$$

by populating a table of values of the function as x approaches 0. **The table should contain at least 5 values**.

For example, you may consider $x = -0.1,\ 0.01,\ -0.001$ etc.

x	$\frac{e^x - 1}{x}$

Fill in your answer:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \boxed{}$$

Recap Problem 2. Sketch the graph of a function f that satisfies the conditions:

$$\lim_{x\to 1} f(x) = 3, \qquad f(1) = 2, \qquad \lim_{x\to 4^+} f(x) = -1, \qquad \lim_{x\to 4^-} f(x) = 1, \qquad f(4) = -1$$

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Day 5. Continuity, Infinite Limits & Limits at Infinity

(based on Sections 2.5, 2.6)

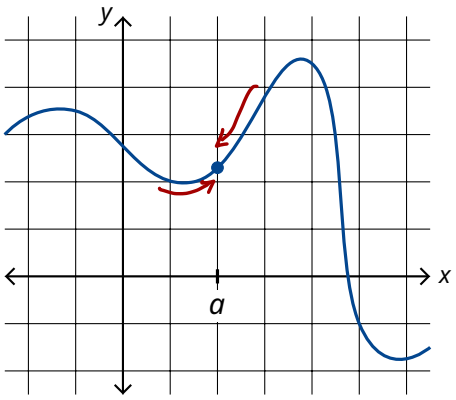
Learning Objectives

- Understand the conceptual ideas of continuity.
- Compute and interpret limits at infinity and infinite limits.
- Identify vertical and horizontal asymptotes using limits at infinity and using infinite limits respectively.

Continuity

Continuity at a Point. Consider a function $f(x)$, and a a number *in its domain*. We say f is **continuous** at a if

$$f(a) = \lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$



Fact. *All our parent functions are continuous.*

Example A. Consider the following function

$$f(x) = \begin{cases} x^2 + a & x < 3 \\ b & x = 3 \\ 2a + x & x > 3 \end{cases}$$

Find a and b such that $f(x)$ is continuous at 3. **(playing in Desmos)**

For continuity, we should have

$$f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

b //

use def^n of $f(x)$ left of 3

use def^n of $f(x)$ right of 3

$3^2 + a = 2a + 3$

$2a + 3 = 9 + a$

$2a - a = 9 - 3$

$a = 6$

Since, we have $b = f(3) = \lim_{x \to 3^-} f(x) = 3^2 + a = 9 + 6$

$\boxed{b = 15}$

Problem 1. Consider the following function

$$f(x) = \begin{cases} 2c + x + e^x & x < 2 \\ e^2 + 3c\sqrt{2x} & x \geqslant 2 \end{cases}$$

Find c such that $f(x)$ is continuous at 2.

For continuity, we should have

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

use defⁿ of $f(x)$ left of 2 — use defⁿ of $f(x)$ right of 2

$2c + 2 + e^2 = e^2 + 3c \cdot \sqrt{2 \cdot 2}$

$2c + 2 + \cancel{e^2} = \cancel{e^2} + 3c \cdot 2$

$6c = 2c + 2$

$4c = 2$

$\boxed{c = \frac{1}{2}}$

Infinite Limits

Infinite Limits and Vertical Asymptotes.

Let f be a function defined on both sides of a, except possibly at a itself. Then,

$\lim_{x \to a} f(x) = \infty$

means that the values of $f(x)$ can be made as large as we please by taking x sufficiently close to a (but not equal to a).

$\lim_{x \to a} f(x) = -\infty$

means that the values of $f(x)$ can be made as *negatively* large as we please by taking x sufficiently close to a (but not equal to a).

Similar definitions can be made to define

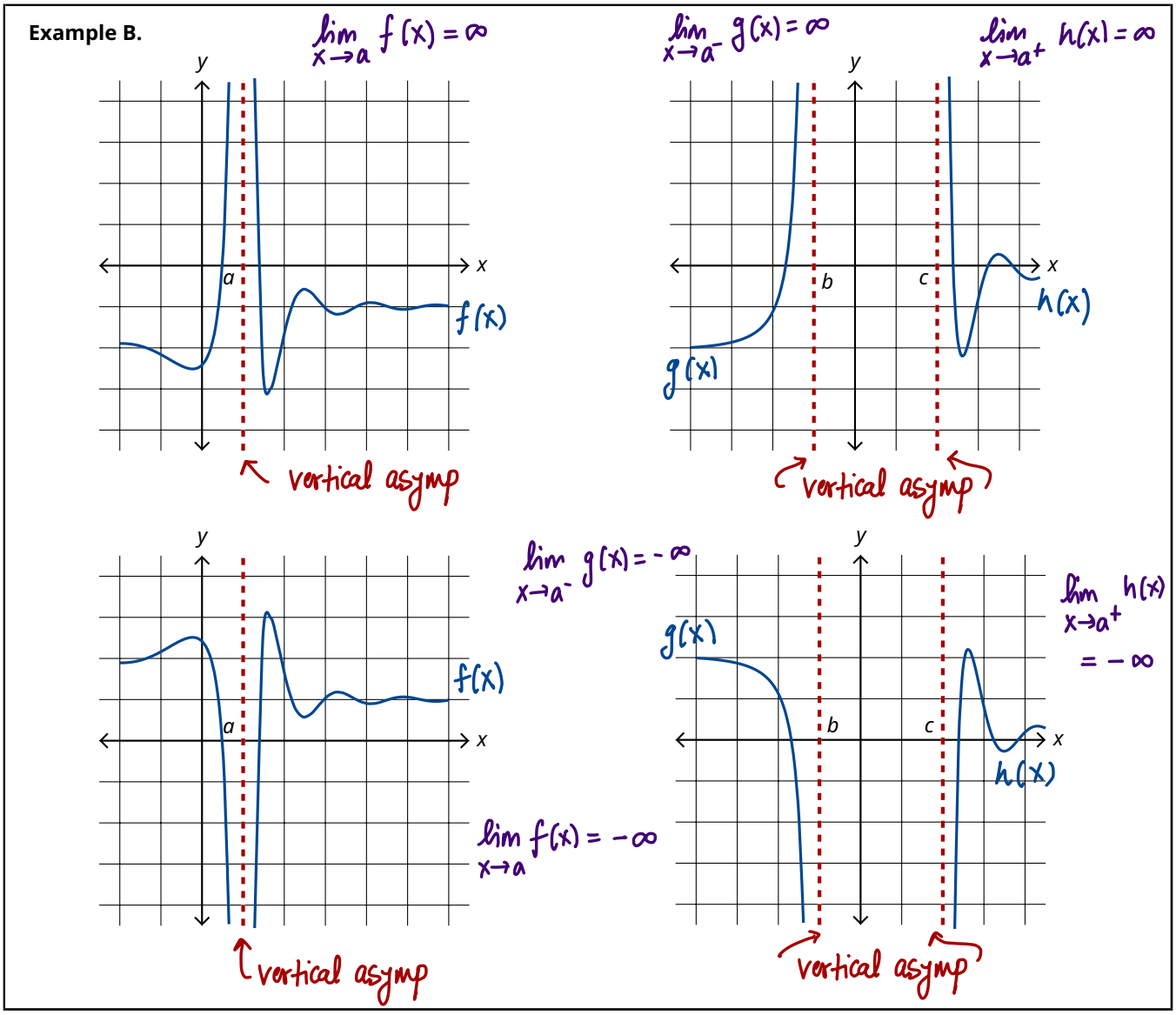
$\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = -\infty$

The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if one of the following statements is true:

$\lim_{x \to a} f(x) = \infty$: $\lim_{x \to a^-} f(x) = \infty$ | $\lim_{x \to a^+} f(x) = \infty$

$\lim_{x \to a} f(x) = -\infty$: $\lim_{x \to a^-} f(x) = -\infty$ | $\lim_{x \to a^+} f(x) = -\infty$

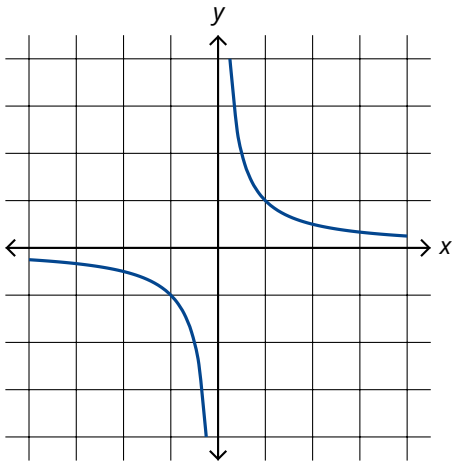
Example B.



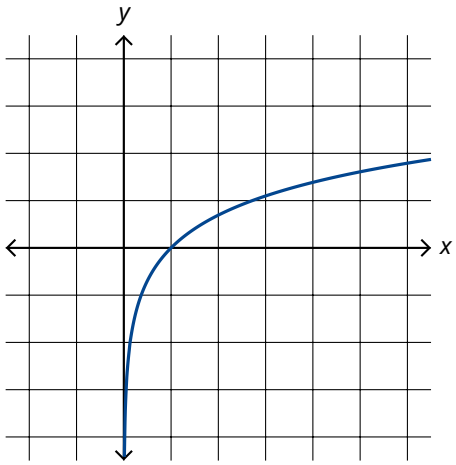
Problem 2. Given below are graphs of some functions. Write down the infinite limits and identify vertical asymptotes.

For example, from the graph of $f(x) = 1/x$ we may deduce that $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$



Graph of $f(x) = 1/x$



Graph of $f(x) = \ln(x)$

$\lim_{x \to 0^+} \ln(x) = -\infty$

Example C. Let us compute the vertical asymptotes of:

(a) $f(x) = \dfrac{x}{x-2}$

When is the denom. = 0?

$x = 2$

↑ vertical asymp

(b) $f(x) = \dfrac{x+3}{x^2-9}$

$x = 3, -3$

not this since also a root of num.

↓

the only vertical asymp

hole at $x = -3$

(c) $f(x) = \dfrac{x^2-9}{x+3}$

No vertical asymp! because deg of num > deg of denom

Vertical Asymptotes & Holes of Rational Functions.

One determines the vertical asymptotes of rational functions $p(x)/q(x)$ as follows:

- If the numerator $p(x)$ degree is $\geqslant$ than the denominator $q(x)$ degree, then there are *no vertical asymptotes*.
- If the numerator $p(x)$ degree is $<$ than the denominator $q(x)$ degree, then there *are vertical asymptotes*:
  - ▷ Find *all* roots of the denominator $q(x)$.

    e.g., in Example C (b), roots of $x^2 - 9$ are -3 and 3.
  - ▷ Determine if any of the roots are also roots of the numerator $p(x)$.

    The rational function has a **hole** here.

    e.g., in Example C (b), -3 is a root of $x + 3$. There is a hole for the rational function at $x = -3$.
  - ▷ The remaining roots is where the rational function has a **vertical asymptote**.

    e.g., in Example C (b), the rational function has a vertical asymptote at $x = 3$.

Limits at Infinity

Limits at Infinity and Horizontal Asymptotes. Let f be a function

defined on some interval (a, ∞). Then, means that the

$\lim_{x \to \infty} f(x) = L$

values of $f(x)$ can be made arbitrarily close to L by making x be sufficiently large.

defined on some interval $(-\infty, a)$. Then, means that

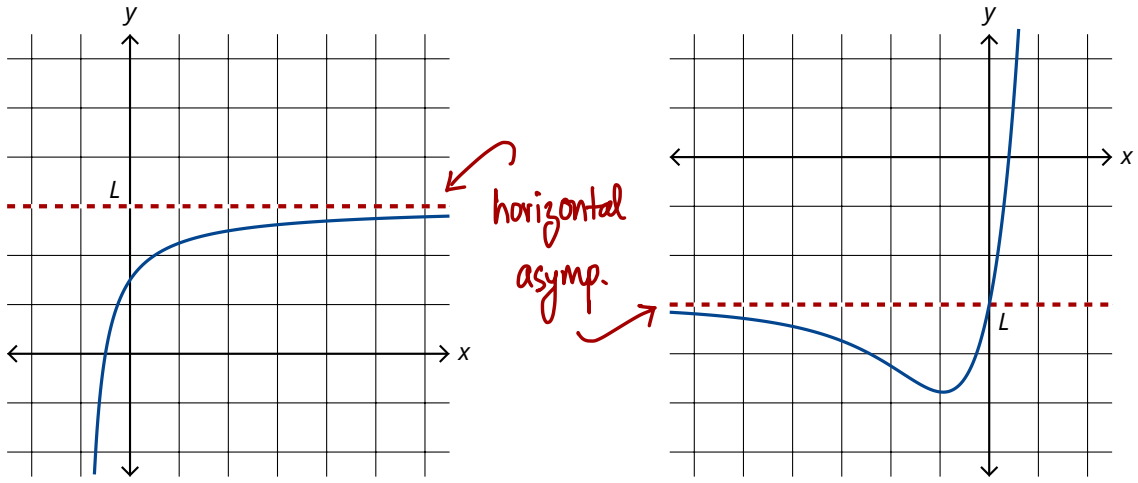
$\lim_{x \to -\infty} f(x) = L$

the values of $f(x)$ can be made arbitrarily close to L by making x be sufficiently negatively large.

The horizontal line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

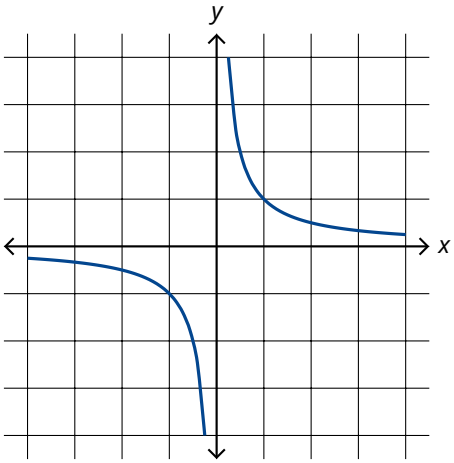
Example D.



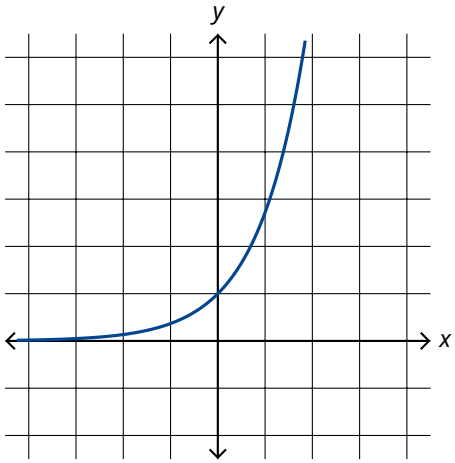
$\lim_{x\to\infty} f(x) = L$ $\lim_{x\to-\infty} f(x) = L$

Problem 3. Given below are graphs of some functions. Write down the limits at infinity and identify horizontal asymptotes.

For example, from the graph of $f(x) = 1/x$ we may deduce that $\lim_{x\to\infty} \frac{1}{x} = 0$.



Graph of $f(x) = 1/x$

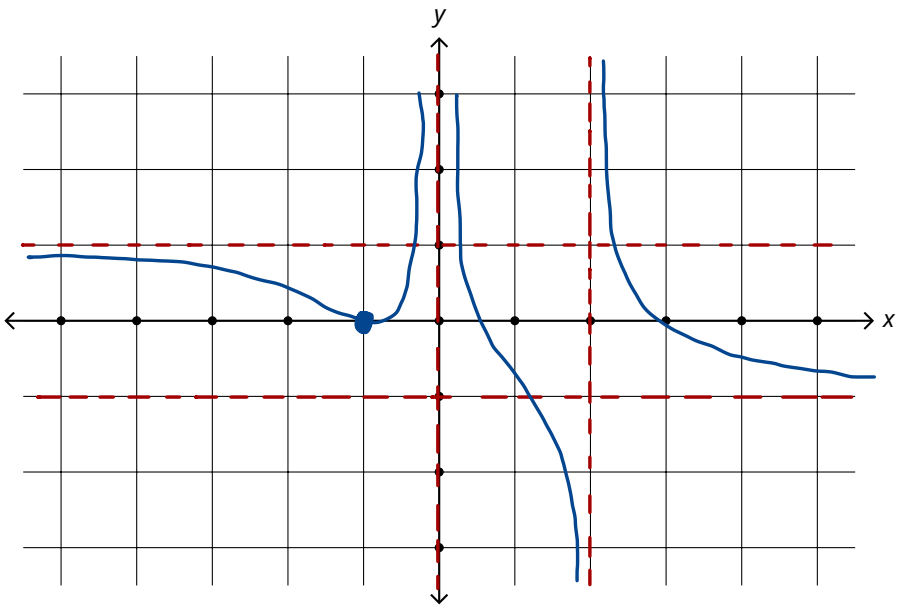


Graph of $f(x) = e^x$

$\lim_{x\to-\infty} \frac{1}{x} = 0$ $\lim_{x\to-\infty} e^x = 0$

Problem 4. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$\lim_{x\to 0} f(x) = \infty$ $\lim_{x\to 2^-} f(x) = -\infty$ $\lim_{x\to 2^+} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = 1$ $\lim_{x\to\infty} f(x) = -1$ $f(-1) = 0$



Important Limit at Infinity. If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

Moreover, if $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Example E. Compute the horizontal asymptotes of the function $f(x) = \dfrac{2x^2 + x}{3x^2 + 4x + 7}$

$$\lim_{x\to\infty} \frac{2x^2+x}{3x^2+4x+7} = \lim_{x\to\infty} \frac{x^2\left(2+\frac{1}{x}\right)}{x^2\left(3+\frac{4}{x}+\frac{7}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3+\frac{4}{x}+\frac{7}{x^2}} = \frac{2+0}{3+4\cdot 0+7\cdot 0} = \frac{2}{3}$$

Same computation for $\lim_{x\to-\infty} f(x)$

Example F. Compute the horizontal asymptotes of the function $f(x) = \dfrac{\sqrt{4x^2 + 2x + 1}}{5x + 3}$

Computing $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+2x+1}}{5x+3}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left(4+2/x+1/x^2\right)}}{x\left(5+3/x\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\cdot\sqrt{4+2/x+1/x^2}}{x\left(5+3/x\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\cdot\sqrt{4+2/x+1/x^2}}{\cancel{x}\left(5+3/x\right)}$$

$$= \frac{\sqrt{4+2\cdot 0+0}}{5+3\cdot 0} = \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2+2x+1}}{5x+3}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(4+2/x+1/x^2\right)}}{x\left(5+3/x\right)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\cdot\sqrt{4+2/x+1/x^2}}{x\left(5+3/x\right)}$$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\cdot\sqrt{4+2/x+1/x^2}}{\cancel{x}\left(5+3/x\right)}$$

$$= \frac{-\sqrt{4+2\cdot 0+0}}{5+3\cdot 0} = -\frac{2}{5}$$

Problem 5. Find each of the following limits.

(a) $\lim_{t\to\infty} \frac{6t^2+t-5}{9-2t^2}$

$$= \lim_{t\to\infty} \frac{\cancel{t^2}\left(6+1/t-5/t^2\right)}{\cancel{t^2}\left(9/t^2-2\right)}$$

$$= \lim_{t\to\infty} \frac{6+1/t-5/t^2}{9/t^2-2}$$

$$= \frac{6+0-5\cdot 0}{9\cdot 0-2}$$

$$= 6/-2 = -3$$

(b) $\lim_{x\to-\infty} \frac{\sqrt{x+3x^2}}{4x-1}$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1/x+3\right)}}{x\left(4-1/x\right)}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\cdot\sqrt{1/x+3}}{x\left(4-1/x\right)}$$

$$= \lim_{x\to-\infty} \frac{-\cancel{x}\cdot\sqrt{1/x+3}}{\cancel{x}\cdot\left(4-1/x\right)} = -\frac{\sqrt{0+3}}{4-0} = -\frac{\sqrt{3}}{4}$$

(c) $\lim_{x\to-\infty} e^{2x}\cos(x)$ (think about applying a squeeze...)

$-1 \le \cos(x) \le 1$

$-e^{2x} \le e^{2x}\cos(x) \le e^{2x}$ (multiply e^{2x})

Observe $\lim_{x\to-\infty} e^{2x} = 0$ (plot the graph of e^{2x} and observe & look at Problem 3)

Similarly $\lim_{x\to-\infty} -e^{2x} = 0$

Applying squeeze theorem $\lim_{x\to-\infty} e^{2x}\cos(x) = 0$

Computing Vertical & Horizontal Asymptotes.

- Compute the degree of the denominator and numerator.

  If you have a polynomial under a square root, ***the degree is taken to be half of the degree*** *of the polynomial under the square root*, e.g., the "degree" of $\sqrt{x^8+1}$ is $8/2 = 4$.

- To compute the vertical asymptote, now repeat the steps laid out on Page 3.

- To compute horizontal asymptotes, we are computing $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We will use

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

- To begin, factor out the highest powers of x in the numerator & denominator.

- Proceed now as in Examples F and E.

- And remember, if you get a $\sqrt{x^2}$, then this is x if your limit is $x \to \infty$ and it is $-x$ if your limit is $x \to -\infty$.

Recap Problems

The aim is to review Day 4 material; so have the filled-in Day 4 notes in front of you as a reference!

Recap Problem 1. Compute the limit:

$$\lim_{x \to 5} \frac{x - 5}{x^2 + 2x - 35}$$

Recap Problem 2. Compute the limit:

$$\lim_{x \to 0} \frac{\frac{1}{x+1} - 1}{x}$$

Name: Deewang

Day 6. Derivative at a Point & as a Function

(based on Sections 2.7, 2.8)

Learning Objectives

- Understand conceptual idea of differentiability.
- Find the derivative at a point *and* as a function.

Derivative at a Point

Derivative at a Point. Let f be a function and a a number in its domain. Then, we say the **derivative** of f at a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

if it exists.

Slope of the Tangent Line. The tangent line to $y = f(x)$ at $(a, f(a))$ is:

the line through $(a, f(a))$ whose slope is $f'(a)$, the derivative of f at a.

Example A. Consider the function $f(x) = 1/x$. Find the equation of the tangent line at $(-1, f(-1))$

Point: $(-1, f(-1)) = (-1, -1)$ and slope: $f'(-1)$

$$f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)}$$

$$= \lim_{x\to -1} \frac{\frac{1}{x}-(-1)}{x+1}$$

$$= \lim_{x\to -1} \frac{1}{x+1}\cdot\left(\frac{1}{x}+1\right)$$

$$= \lim_{x\to -1} \frac{1}{\cancel{x+1}}\cdot\frac{\cancel{1+x}}{x}$$

$$= \lim_{x\to -1} \frac{1}{x}$$

$$= -1$$

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{-1+h}-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{1}{h-1}+1\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{1+(h-1)}{h-1}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\cdot\frac{\cancel{h}}{h-1}$$

$$= \lim_{h\to 0} \frac{1}{h-1} = \frac{1}{0-1} = -1$$

Equation of the tangent line: $y-(-1)=-1\cdot(x-(-1)) \rightsquigarrow y=-x-2$

Problem 1. Compute the slope of the tangent line to the graph of $f(x)=x^2+1$ at the point $(1,2)$ in two ways below. ($f(1)$)

(a) First, as the limit: $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$$= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1}\frac{x^2+1-2}{x-1}$$

$$= \lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} x+1$$

$$= 1+1$$

$$= 2$$

(b) Second, as the limit: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)^2+1-2}{h}$$

$$= \lim_{h\to 0}\frac{(1+2h+h^2)-1}{h}$$

$$= \lim_{h\to 0}\frac{2h+h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2+h)}{h}$$

$$= \lim_{h\to 0} 2+h$$

$$= 2$$

(c) Find the equation of the tangent line to $y=f(x)$ at $(1,2)$.

$y-2 = 2\cdot(x-1)$ ($2 = f'(1)$) $\rightsquigarrow$ $\boxed{y=2x}$

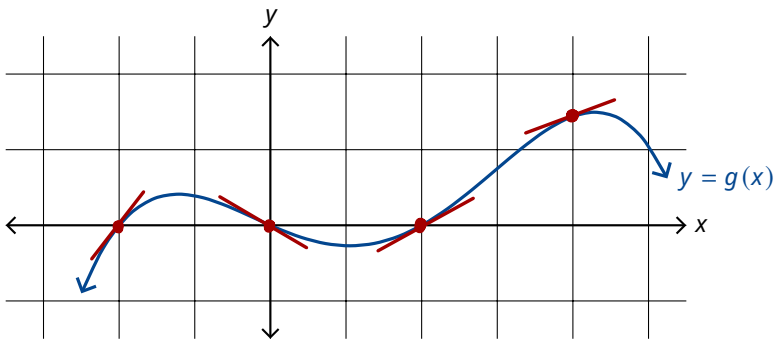
(d) Plot the tangent line you found above and $y=f(x)$ on Desmos. Zoom in on your graph and verify that the graphs begin to coincide as you zoom in.

Problem 2. For the function g whose graph is shown below, arrange the following numbers in increasing order:

$$0, \quad g'(-2), \quad g'(0), \quad g'(2), \quad g'(4)$$

Instructions: draw mini tangent lines at the appropriate points on the graph; remember the derivative is their slope.

I've sketched little tangent lines whose slopes are the desired derivatives.



We order them by steepness

$$g'(0) < 0 < g'(2) < g'(4) < g'(-2)$$

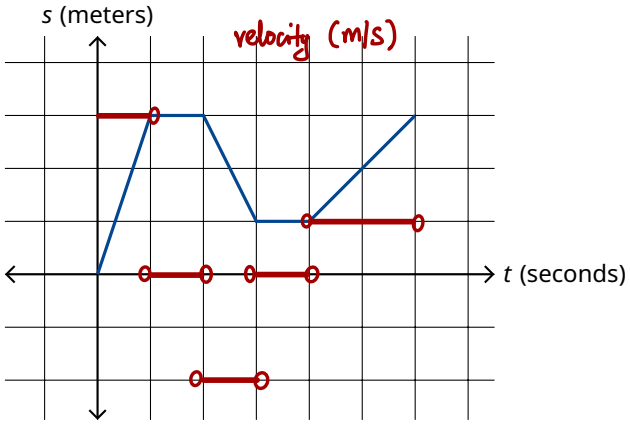
Instantaneous Rate of Change. If $s(t)$ is a distance function.

Then the instantaneous velocity at $t = a$ is $s'(a)$, the derivative of s at a.

Problem 3. Shown below is a graph depicting the position s of a particle moving along a line. Sketch a graph of the velocity of the particle, using the same axes below.

Important Note. *If $(a, s(a))$ is on a straight line, then $s'(a)$ is the slope of the line the point is on.*

We compute the slopes of each line segment.



Derivative as a Function

Derivative as a Function. Let f be a function. Its **derivative as a function in** x is provided the limit exists.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

That is, the *domain of f' are those x such that the limit above exists*.

Example B. Find the derivative of $f(x) = \sqrt{x}$. What is its domain? Find the equation of the tangent line at $x = 4$.

1. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \times \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \dfrac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h\cdot(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \dfrac{(x+h)-x}{h\cdot(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}\cdot(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{1}{\sqrt{x+0}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

2. $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

3. Eq^n of tangent line: $y - f(4) = f'(4)(x-4)$

$y - 2 = \frac{1}{4}(x-4) \rightsquigarrow y = \frac{1}{4}x + 1$

Example C. What is the derivative of $f(x) = |x|$ and what is its domain?

$$f(x) = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases} \rightsquigarrow f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

What about $f'(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \dfrac{f(h)}{h}$ D.N.E.

$\lim_{h\to 0^-} \dfrac{f(h)}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1 \neq \lim_{h\to 0^+} \dfrac{f(h)}{h} = \lim_{h\to 0^+} \dfrac{h}{h} = 1$

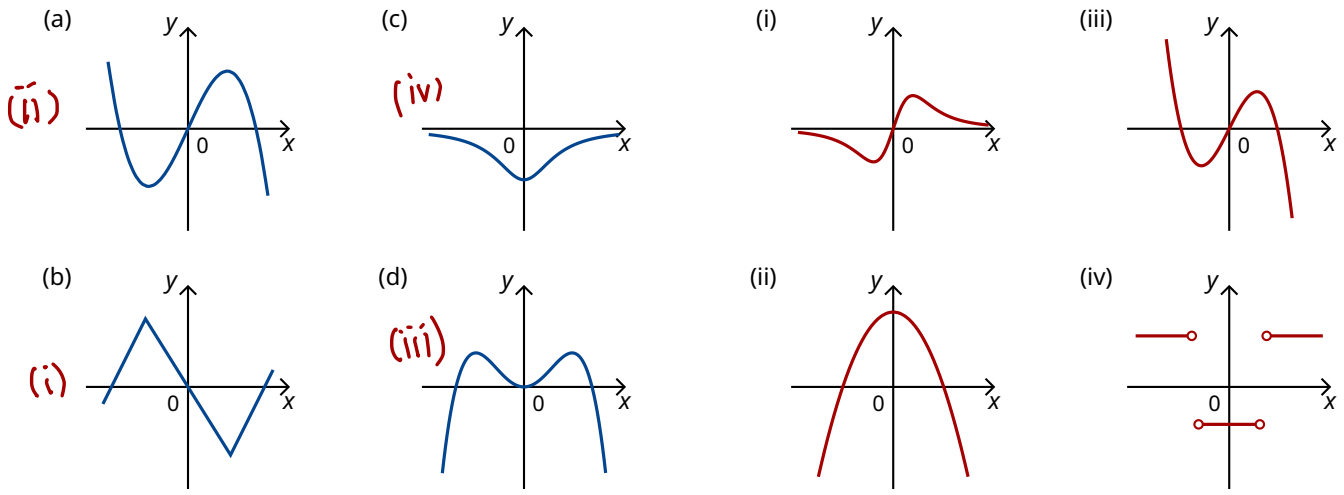
Types of Non-differentiability. (Desmos)

The derivative of $f(x)$ at a point $x = a$ **does not exist** if

- there is a *"sharp corner"* at a in the graph of the function;
- there is a *vertical tangent line* at a in the graph of the function;
- the function is *not continuous* at a.

Problem 4. Match the graph of each function (a) – (d) with the graph of its derivative (i) – (iv).

Strategy. Look at the derivative graphs on the right and see where the they are negative/positive/zero; compare with slopes of mini tangent lines on the function graphs on the left.



Leibniz Notation & Higher-order Derivatives. There is another notation for the derivative of a function f.

The notation emphasises the independent variable: if $y = f(x)$, then we write

$$f'(x) = \frac{dy}{dx}$$

For example, if the independent variable is t, i.e., $y = f(t)$, then we may write:

$$f'(t) = \frac{dy}{dt}$$

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function, so f' may have a derivative of its own, denoted

$$f''(x) = \frac{d^2y}{dx^2}$$

This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

More generally, the ***n*-th derivative** of f, obtained by differentiating f n-many times, is denoted

$$f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Velocity & Acceleration. The *first derivative* of a distance function is **velocity**.

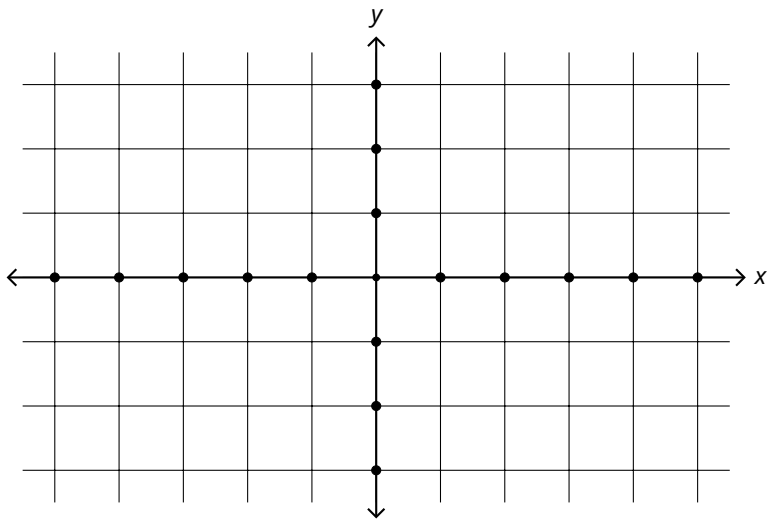
The *second derivative* of a distance function, or the *first derivative* of a velocity function, is **acceleration**.

Recap Problems

The aim is to review Day 5 material; so have the filled-in Day 5 notes in front of you as a reference!

Recap Problem 1. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$\lim_{x\to 2} f(x) = -\infty$ $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$ $f(2) = 0$ $\lim_{x\to -2} f(x) = 1$, but $f(-2)$ is undefined



Recap Problem 2. Compute the limit:

$$\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6}$$

Name: Deewang

Day 7. Shortcut Rules and the Product Rule

(based on Sections 3.1, 3.2)

Learning Objectives

- Use shortcut rules (Power, Constant, Sum/Difference) to compute derivatives efficiently.
- Differentiate exponential functions with base e and general base a.
- Apply the Product Rule to functions that are products of two differentiable functions.

Leibniz Notation & Higher-order Derivatives. There is another notation for the derivative of a function f.

The notation emphasises the independent variable: if $y = f(x)$, then we write

$$f'(x) = \frac{dy}{dx}$$

For example, if the independent variable is t, i.e., $y = f(t)$, then we may write:

$$f'(t) = \frac{dy}{dt}$$ "differentiating with respect to t"

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function, so f' may have a derivative of its own, denoted

$$f''(x) = \frac{d^2y}{dx^2}$$

This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

More generally, the ***n*-th derivative** of f, obtained by differentiating f n-many times, is denoted

$$f'''(x) = \frac{d^3y}{dx^3} \rightsquigarrow f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Velocity & Acceleration. The *first derivative* of a distance function is **velocity**.

The *second derivative* of a distance function, or the *first derivative* of a velocity function, is **acceleration**.

Shortcut Rules

The Power Rule.

For the function $f(x) = x^r$, where r is any real number,

$$\frac{d}{dx}(f(x)) = f'(x) = r \cdot x^{r-1}$$

The Constant Multiple Rule.

For any differentiable function $f(x)$ and c a constant,

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{d}{dx}(f(x))$$

Example A. Use the power rule to find the first *and second* derivatives of the following functions:

(a) $f(x) = 5x^{100}$

$$f'(x) = \frac{d}{dx}(5x^{100})$$

$$= 5 \cdot \frac{d}{dx}(x^{100})$$

$$= 5 \cdot (100x^{100-1})$$

$$= 500x^{99}$$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$= \frac{d}{dx}(500x^{99})$$

$$= 500\frac{d}{dx}(x^{99})$$

$$= 500(99x^{99-1})$$

$$= 49500x^{98}$$

(b) $y = \frac{6}{\sqrt[4]{x}} = 6x^{-1/4}$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^{-1/4})$$

$$= 6 \cdot \frac{d}{dx}(x^{-1/4})$$

$$= 6 \cdot \left(-\frac{1}{4} \cdot x^{-1/4-1}\right)$$

$$= -\frac{6}{4} \cdot x^{-5/4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{6}{4} \cdot x^{-5/4}\right)$$

$$= -\frac{6}{4}\frac{d}{dx}(x^{-5/4})$$

$$= -\frac{6}{4} \cdot \left(-\frac{5}{4}x^{-5/4-1}\right)$$

$$= \frac{15}{8}x^{-9/4}$$

Sum/Difference Rules. Let f and g be differentiable functions. Then,

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

Example B. Using a combination of the rules we have introduced above, find

(a) $\frac{d}{dx}\left(x^8 - 12x^5 + 5\right)$

$$\frac{d}{dx}(x^8) - \frac{d}{dx}(12x^5) + \frac{d}{dx}(5)$$

$$= \frac{d}{dx}(x^8) - 12\frac{d}{dx}(x^5) + \frac{d}{dx}(5)$$

$$= 8x^{8-1} - 12(5x^{5-1}) + 0$$

$$= 8x^7 - 60x^4$$

(b) the derivative of $h(\theta) = \theta\left(3\theta^4 - \frac{5}{\sqrt{\theta}} + 7\theta^{-2}\right)$

$$= 3\theta^5 - \frac{5\theta}{\theta^{1/2}} + 7\theta^{1-2}$$

$$= 3\theta^5 - 5\theta^{1/2} + 7\theta^{-1}$$

$$h'(\theta) = (3\theta^5)' - (5\theta^{1/2})' + (7\theta^{-1})'$$

$$= 3\cdot(\theta^5)' - 5\cdot(\theta^{1/2})' + 7\cdot(\theta^{-1})'$$

$$= 3(5\theta^{5-1}) - 5\left(\tfrac{1}{2}\theta^{1/2-1}\right) + 7(-1\cdot\theta^{-1-1})$$

$$= 15\theta^4 - \frac{5}{2}\theta^{-1/2} - 7\theta^{-2}$$

$$= 15\theta^4 - \frac{5}{2\sqrt{\theta}} - \frac{7}{\theta^2}$$

Derivatives of Exponential Functions.

For the function $f(x) = a^x$, where $a > 0$,

$$\frac{d}{dx}(a^x) = a^x \cdot \ln(a)$$

In particular, for the function e^x

$$\frac{d}{dx}(e^x) = e^x \cdot \ln(e) = e^x$$

Problem 1. Differentiate each of the functions below using the differentiation "shortcut rules" above. Note at each step which rule you are using (if any).

(a) $f(x) = e^x + \sqrt[3]{x} + e^3$

$$f'(x) = (e^x)' + (x^{1/3})' + (e^3)'$$

$$= e^x + \frac{1}{3}x^{1/3-1} + 0 = e^x + \frac{1}{3}x^{-2/3}$$

For the following: **first write them as a sum of functions**, *either by splitting the fraction or multiplying everything through*.

(b) $f(x) = \dfrac{\sqrt{x} + x^4}{x^2}$

$= \dfrac{\sqrt{x}}{x^2} + \dfrac{x^4}{x^2} = \dfrac{x^{1/2}}{x^2} + x^2$

$= x^{1/2-2} + x^2$

$f(x) = x^{-3/2} + x^2$

$f'(x) = -\dfrac{3}{2} x^{-3/2-1} + 2x^{2-1}$

$= -\dfrac{3}{2} x^{-5/2} + 2x$

(c) $g(t) = t^2\left(3t + \dfrac{1}{t^3}\right)$

$= 3t^3 + \dfrac{1}{t} = 3t^3 + t^{-1}$

$g'(t) = 3(3t^{3-1}) + (-1 \cdot t^{-1-1})$

$= 9t^2 - \dfrac{1}{t^2}$

Problem 2. Find a quadratic polynomial $P(x) = ax^2 + bx + c$ such that $P(2) = 5$, $P'(2) = 3$ and $P''(2) = 2$.

$P(x) = ax^2 + bx + c \longrightarrow 5 = P(2) = a \cdot (2)^2 + b \cdot 2 + c$

$= 4a + 2b + c$

$= 4 - 2 + c$

$= 2 + c$

$P'(x) = 2ax + b$

$P''(x) = 2a$

$2 = P''(2) = 2a \rightsquigarrow \boxed{a = 1}$

$\rightsquigarrow \boxed{c = 3}$

$3 = P'(2) = 2a \cdot 2 + b = 4a + b = 4 + b$

$\rightsquigarrow \boxed{b = -1}$

$\boxed{P(x) = x^2 - x + 3}$

The Product Rule

Question. How do we differentiate functions like xe^x?

Warning: Incorrect Computation/Rule. Consider the function $f(x) = x^5$

We know that

$$\frac{d}{dx}\left(x^5\right) = 5x^4$$

What about the following computation:

$$\frac{d}{dx}\left(x^5\right) = \frac{d}{dx}\left(x^2 \cdot x^3\right) \stackrel{?}{=} \frac{d}{dx}\left(x^2\right) \cdot \frac{d}{dx}\left(x^3\right) = 2x \cdot 3x^2 \ ?!$$

The formula $\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$ **is wrong.** The correct formula is given by the **product rule** ↓

Product Rule. $f(x)$ and $g(x)$ are differentiable functions. Then,

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$$

Example C. Find the derivative of $h(x) = xe^x$.

Step 1. First find two functions that above function is a product of:

$$a(x) = x, \quad b(x) = e^x \quad \rightsquigarrow \quad h(x) = a(x) \cdot b(x)$$

Step 2. Write down the product formula to identify what needs to be computed:

$$h'(x) = a'(x)\,b(x) + a(x)\,b'(x)$$

Step 3. Do the relevant computations using shortcut derivative rules:

$$a'(x) = 1, \qquad b'(x) = e^x$$

Step 4. Package everything and write down the final answer, and make any simplification you wish to make:

$$h'(x) = a'(x)\,b(x) + a(x)\,b'(x) = e^x + xe^x$$

Problem 3. Compute the derivative of $k(t) = (t^2 + e^t)\sqrt{t}$.

Step 1. First find two functions that above function is a product of:

$$a(t) = t^2 + e^t \quad , \quad b(t) = \sqrt{t} \quad \rightsquigarrow \quad k(t) = a(t) \cdot b(t)$$

Step 2. Write down the product formula to identify what needs to be computed:

$$k'(t) = a'(t)\, b(t) + a(t)\, b'(t)$$

Step 3. Do the relevant computations using shortcut derivative rules:

$$a'(t) = 2t + e^t, \quad b'(t) = (t^{1/2})' = \frac{1}{2} t^{-1/2} = \frac{1}{2\sqrt{t}}$$

Step 4. Package everything and write down the final answer, and make any simplification you wish to make:

$$k'(t) = a'(t)\, b(t) + a(t)\, b'(t) = (2t + e^t)\sqrt{t} + (t^2 + e^t) \cdot \frac{1}{2\sqrt{t}}$$

Problem 4. Find the equation of the tangent line to the function $y = x^3 e^x$ at the point $(1, e)$.

$$\frac{dy}{dx} = (x^3)' \cdot e^x + x^3 \cdot (e^x)' = 3x^2 e^x + x^3 e^x$$

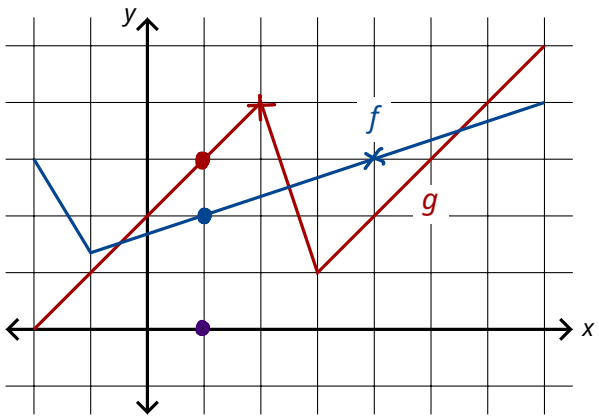
$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \cdot 1^2 \cdot e^1 + 1^3 \cdot e^1 = 3e + e = 4e$$

notation for $\frac{dy}{dx}$ at $x=1$; analogous with $f'(1)$

Eqn of tangent line: $y - e = 4e\,(x - 1)$

$$\rightsquigarrow \boxed{y = 4ex - 3e}$$

Problem 5. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$.



(a) From the graph, compute $f(1)$, $g(1)$, $f'(1)$ and $g'(1)$.

$f(1) = 2$ $\quad f'(1) =$ slope of the line $(1, f(1))$ is on $= \frac{3-2}{4-1} = \frac{1}{3}$

$g(1) = 3$ $\quad g'(1) =$ slope of the line $(1, g(1))$ is on $= \frac{4-3}{3-1} = 1$

(b) Now compute $u'(1)$ by writing it in terms of $f(1)$, $g(1)$, $f'(1)$ and $g'(1)$.

$$u'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$u'(1) = f'(1)\,g(1) + f(1)\,g'(1)$$

$$= \frac{1}{3} \cdot 3 + 2 \cdot 1$$

$$= 3$$

Recap Problems

The aim is to review Day 6 material; so have the filled-in Day 6 notes in front of you as a reference!

Recap Problem 1. For the function $f(x) = \sqrt{x+2}$, find $f'(2)$ by computing the limit:

(**Hint.** Substitute $f(x)$ and $f(2)$ and then rationalise the numerator)

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} =$$

Recap Problem 2. For the function $f(x) = x^2 + x$, find $f'(x)$ by computing the limit:

(**Hint.** Substitute $f(x+h)$ and $f(x)$ and expand $(x+h)^2$ to cancel terms; your main aim is to cancel out the h in the denominator)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

Name: Deewang

Day 8. The Quotient Rule and Trig. Differentiation

(based on Sections 3.2, 3.3)

Learning Objectives

- Apply the Quotient Rule to functions that are quotients of two differentiable functions.
- Differentiate functions involving sine, cosine, tangent, and other trigonometric functions.
- Interpret trigonometric derivatives in the context of physical models.

The Quotient Rule

Warning:

The formula $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)}{g'(x)}$ **is wrong.**

The correct formula is given by the **quotient rule** ↓

Quotient Rule. $f(x)$ and $g(x) \neq 0$ are differentiable functions. Then,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example A. Find the derivative of $g(x) = \frac{1+\sqrt{x}}{2 \cdot 3^x}$. $= \frac{a(x)}{b(x)}$

Step 1. First find two functions that the above function is a quotient of: $a(x) = 1+\sqrt{x}$, $b(x) = 2 \cdot 3^x$

Step 2. Write down the quotient formula to identify what needs to be computed:

$$g'(x) = \frac{a'(x)\,b(x) - a(x)\,b'(x)}{b(x)^2}$$

Step 3. Do the relevant computations using shortcut derivative rules:

$$a'(x) = \frac{1}{2\sqrt{x}}, \quad b'(x) = 2 \cdot 3^x \ln(3)$$

Step 4. Package everything and write down the final answer, and make any simplification you wish to make:

$$g'(x) = \frac{a'(x)\,b(x) - a(x)\,b'(x)}{b(x)^2} = \frac{\frac{1}{2\sqrt{x}} \cdot 2 \cdot 3^x - (1+\sqrt{x})\,2 \cdot 3^x \cdot \ln(3)}{(2 \cdot 3^x)^2}$$

Problem 1. Compute the derivatives of:

(a) $h(r) = \dfrac{\sqrt[3]{r}}{r-3} = \dfrac{r^{1/3}}{r-3}$

$$h'(r) = \frac{(r^{1/3})' \cdot (r-3) - r^{1/3}(r-3)'}{(r-3)^2}$$

$$= \frac{\frac{1}{3} \cdot r^{-2/3}(r-3) - r^{1/3}}{(r-3)^2}$$

(b) $k(t) = \dfrac{10t^2}{e^t}$

$$k'(t) = \frac{(10t^2)' \cdot e^t - 10t^2 \cdot (e^t)'}{(e^t)^2}$$

$$= \frac{20t \cdot e^t - 10t^2 \cdot e^t}{(e^t)^2}$$

Problem 2. Find the equation of the tangent line to the function $y = \dfrac{x^2}{1+x}$ at the point $(1, 0.5)$.

$$\frac{dy}{dx} = \frac{(x^2)' \cdot (1+x) - x^2 \cdot (1+x)'}{(1+x)^2} = \frac{2x \cdot (1+x) - x^2}{(1+x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2 \cdot 1 \cdot (1+1) - 1^2}{(1+1)^2} = \frac{3}{4}$$

Eqn: $y - 0.5 = 0.75(x-1)$

Example B. Find the derivative of $f(x) = \dfrac{4xe^x}{1+\sqrt{x}}$ ($\leftarrow a(x)$, $\leftarrow b(x)$)

$b(x) = 1+\sqrt{x} \rightsquigarrow b'(x) = \dfrac{1}{2\sqrt{x}}$

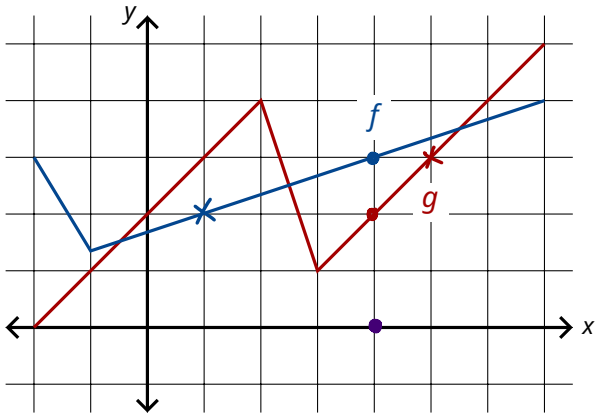
$a(x) = 4x \cdot e^x \rightsquigarrow a'(x) = (4x)' \cdot e^x + 4x \cdot (e^x)'$

$= 4e^x + 4xe^x$

$$f'(x) = \frac{a'(x)\,b(x) - a(x)\,b'(x)}{(b(x))^2}$$

$$= \frac{(4e^x + 4xe^x) \cdot (1+\sqrt{x}) - 4xe^x \cdot \frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$$

Problem 3. If f and g are the functions whose graphs are shown, let $v(x) = \dfrac{f(x)}{g(x)}$.



(a) From the graph, compute $f(4)$, $g(4)$, $f'(4)$ and $g'(4)$.

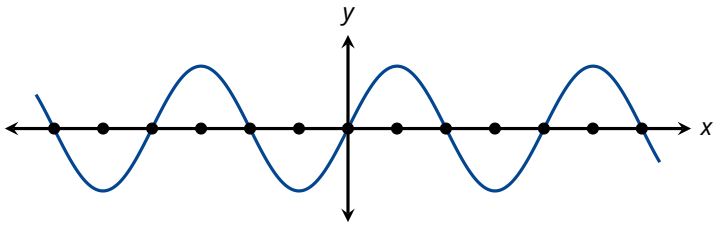
$f(4) = 3$ $\quad f'(4) =$ slope of the line $(4, f(4))$ is on $= \dfrac{3-2}{4-1} = \dfrac{1}{3}$

$g(4) = 2$ $\quad g'(4) =$ slope of the line $(4, g(4))$ is on $= \dfrac{2-3}{4-5} = 1$

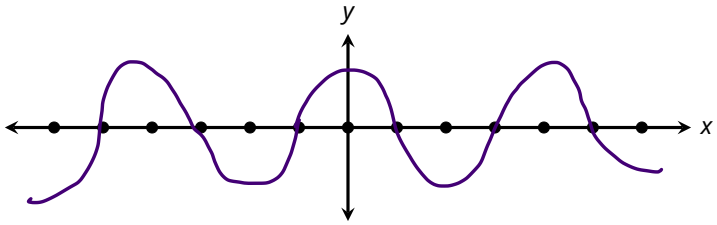
(b) Now compute $v'(4) =$ $\dfrac{f'(4)\,g(4) - f(4)\,g'(4)}{g(4)^2} = \dfrac{\frac{1}{3}\cdot 2 - 3\cdot 1}{2^2} = -\dfrac{7}{12}$

Derivatives of Trigonometric Functions.

This is the graph of $y = \sin(x)$



Let use this to draw the graph of $\dfrac{dy}{dx} = (\sin(x))'$



Derivatives of Trigonometric Functions I. • $\dfrac{d}{dx}(\sin(x)) =$ $\cos(x)$ • $\dfrac{d}{dx}(\cos(x)) =$ $-\sin(x)$

Example C. Let us compute the derivative of $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ ($\sin(x) \to f(x)$, $\cos(x) \to g(x)$)

$$\frac{d}{dx}(\tan x) = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x - \sin x\,(-\sin x)}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$$= (\sec x)^2$$

Problem 4. Find the derivative of $f(x) = \dfrac{1}{\sin(x)}$. ($1 \to a(x)$, $\sin(x) \to b(x)$)

$$f'(x) = \frac{(1)' \cdot \sin x - 1 \cdot (\sin x)'}{(\sin x)^2}$$

$$= \frac{0 \cdot \sin x - 1 \cdot (\cos x)}{(\sin x)^2} = -\frac{\cos x}{(\sin x)^2}$$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cdot \cot x$$

Derivatives of Trigonometric Functions II. The remaining trigonometric functions are

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$$

- $\dfrac{d}{dx}(\tan(x)) =$ $(\sec x)^2$
- $\dfrac{d}{dx}(\cot(x)) =$ $-(\csc x)^2$
- $\dfrac{d}{dx}(\csc(x)) =$ $-\csc x \cdot \cot x$
- $\dfrac{d}{dx}(\sec(x)) =$ $\sec x \cdot \tan x$

Problem 5. Compute the derivatives of:

(a) $f(\theta) = \tan(\theta) - 4\sin(\theta)$

$$f'(\theta) = \frac{d}{d\theta}(\tan\theta) - 4\cdot\frac{d}{d\theta}(\sin\theta)$$

$$= (\sec\theta)^2 - 4\cdot\cos\theta$$

(b) $g(\theta) = \sin(\theta)\cos(\theta)$

$$g'(\theta) = \frac{d}{d\theta}(\sin\theta)\cdot\cos\theta + \sin\theta\cdot\frac{d}{d\theta}(\cos\theta)$$

$$= \cos\theta\cdot\cos\theta + \sin\theta(-\sin\theta)$$

$$= (\cos\theta)^2 - (\sin\theta)^2$$

Problem 6. Compute the derivatives of:

(a) $g(t) = \dfrac{\cos(t)}{1-\sin(t)}$

$$g'(t) = \frac{(\cos t)'\cdot(1-\sin t) - \cos t\cdot(1-\sin t)'}{(1-\sin t)^2}$$

$$= \frac{(-\sin t)\cdot(1-\sin t) - \cos t\,(-\cos t)}{(1-\sin t)^2}$$

$$= \frac{-\sin t + (\sin t)^2 + (\cos t)^2}{(1-\sin t)^2}$$

$$= \frac{-\sin t + 1}{(1-\sin t)^2}$$

$$= \frac{1}{1-\sin t}$$

(b) $f(t) = e^t\sin(t) + t^2\cos(t)$

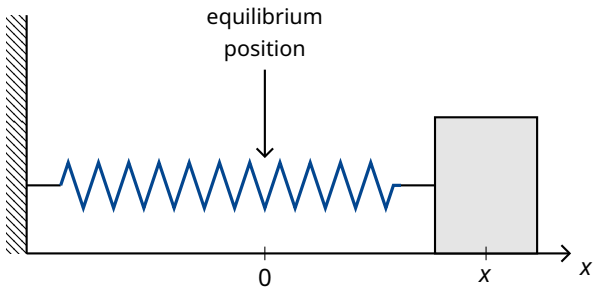
$$f'(t) = (e^t\cdot\sin t)' + (t^2\cdot\cos t)'$$

$$= (e^t)'\cdot\sin t + e^t\cdot(\sin t)' + (t^2)'\cdot\cos t + t^2\cdot(\cos t)'$$

$$= e^t\cdot\sin t + e^t\cos t + 2t\cdot\cos t - t^2\cdot\sin t$$

$$= (e^t - t^2)\cdot\sin t + (e^t + 2t)\cdot\cos t$$

Problem 7. The figure below shows a mass attached to a spring which oscillates back and forth on a smooth surface.



Its motion is given by

$$x(t) = 8\sin(t),$$

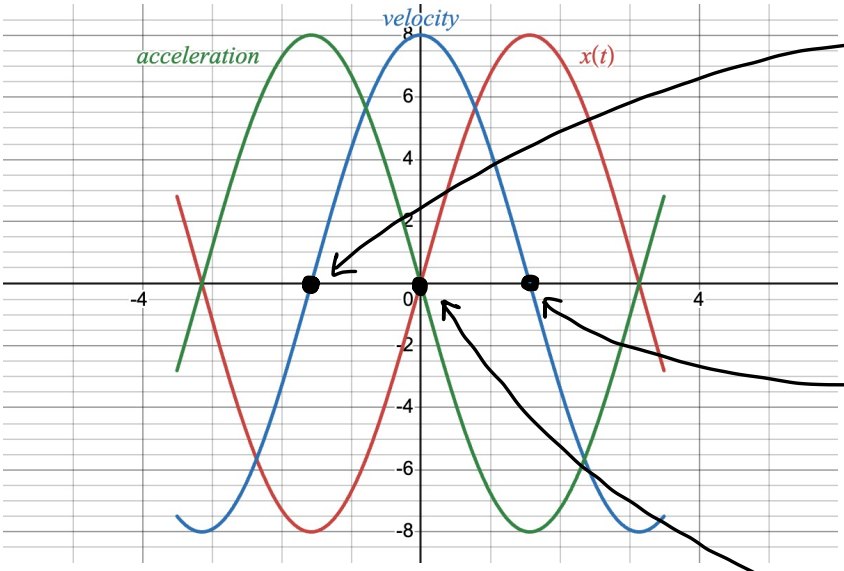
where x is measured in centimeters and t in seconds.

(a) Find the velocity and acceleration of the mass at time t.

velocity $= x'(t) = 8\cos t$

acceleration $= x''(t) = \frac{d}{dt}(x'(t)) = -8\sin t$

(b) Plot the position, velocity and acceleration on the same axes. Note points on the graph where the mass is at its left-most position, its right-most position, and where it is passing the equilibrium position. What are the possibly accelerations and velocities at these noted points?



mass at its left-most:

t where x(t) is most negative

velocity = 0 acceleration most positive

mass at its right-most:

t where x(t) is most positive

velocity = 0 acceleration most negative

passing the equilibrium point

velocity = maximum acceleration = 0

Recap Problems

The aim is to review Day 7 material; so have the filled-in Day 7 notes in front of you as a reference!

Recap Problem 1. Compute the derivative of $5\sqrt[4]{x} - 7^x + 11$.

Make sure you have reviewed derivatives of general exponential functions for computing the derivative of 7^x

Recap Problem 2. Compute the derivative of the following function by applying the product rule *once*

$$4e^x \cdot x\sqrt{x}$$

First re-write $x\sqrt{x}$ as a power function, and *then apply the product rule*.

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Day 9. The Chain Rule

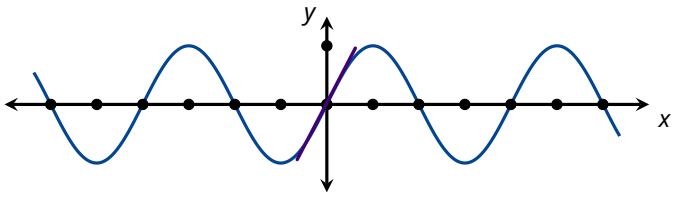
(based on Sections 3.2, 3.3)

Learning Objectives

- Identify composite functions and apply the Chain Rule to find their derivatives.
- Combine the Chain Rule with the product and quotient rules.
- Use graphical information to compute the derivative of a composite function.

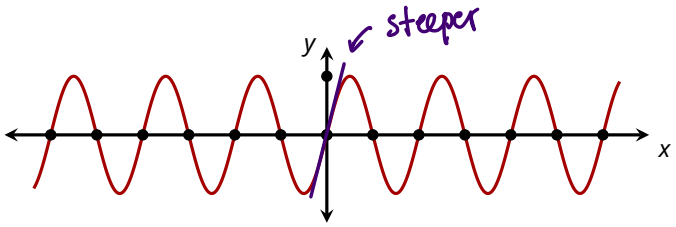
The Chain Rule

This is the graph of $y = \sin(x)$



and $\frac{dy}{dx} = \cos(x)$

This is the graph of $y = \sin(2x)$



and $\frac{dy}{dx} =$ $2\cos(2x)$

Example A. Suppose we are hiking and know the rate of change of temperature with respect to the change in altitude, and the change in altitude with respect to time. How could we find the rate of change in temperature with respect to the change in time?

Temperature K is a function of altitude a. $K = f(a)$

Altitude a is a function of time t. $a = g(t)$

$$K = f(a) = f(g(t)) \rightsquigarrow \frac{dK}{dt} = \left(f(g(t))\right)'$$

$$= f'(g(t)) \cdot g'(t) = f'(a) \cdot g'(t) = \frac{dK}{da} \cdot \frac{da}{dt}$$

Chain Rule. If f and g are differentiable functions such that we can consider their composition $f(g(x))$. Then,

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, write $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example B. Let us compute the derivative of $\sqrt{2x + e^x}$.

Step 1. First find two functions that above function is a composition of: $f(x) = \sqrt{x}$, $g(x) = 2x + e^x$

$$\sqrt{2x+e^x} = f(g(x))$$

Step 2. Write down the chain rule to identify what needs to be computed:

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

Step 3. Do the relevant computations using shortcut derivative rules:

$g'(x) = 2 + e^x$ $\quad f'(x) = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad f'(g(x)) = \frac{1}{2\sqrt{2x+e^x}}$

Step 4. Package everything and write down the final answer, and make any simplification you wish to make:

$$\frac{d}{dx}\left(\sqrt{2x+e^x}\right) = f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$= \frac{2+e^x}{2\sqrt{2x+e^x}}$$

Example C. Let us compute the derivatives of:

(a) $h(x) = 2e^{x^2+1}$

$= f(g(x))$

where $f(x) = 2e^x$, $g(x) = x^2+1$

$g'(x) = 2x$, $f'(x) = 2e^x$

$f'(g(x)) = 2e^{g(x)} = 2e^{x^2+1}$

$h'(x) = f'(g(x)) \cdot g'(x) = 2e^{x^2+1} \cdot 2x$

(a) $(f(x))^2$

$= a(f(x))$ where $a(x) = x^2$

$a'(x) = 2x \rightsquigarrow a'(f(x)) = 2f(x)$

$\frac{d}{dx}\left((f(x))^2\right) = a'(f(x)) \cdot f'(x)$

$= 2f(x) \cdot f'(x)$

Problem 1. Compute the derivatives of:

(a) $f(t) = \sin(t^2)$ (outside: $\sin(t)$, inside: t^2)

$a(t) = \sin(t), \quad b(t) = t^2$

$f(t) = a(b(t))$

$f'(t) = a'(b(t)) \cdot b'(t)$

$= \cos(t^2) \cdot 2t$

(b) $k(\theta) = 3e^{2\sin(\theta)}$ (outside: $3e^{\theta}$, inside: $2\sin(\theta)$)

$a(\theta) = 3e^{\theta}, \quad b(\theta) = 2\sin(\theta)$

$k(\theta) = a(b(\theta))$

$k'(\theta) = a'(b(\theta)) \cdot b'(\theta)$

$= 3e^{2\sin\theta} \cdot 2\cos\theta$

(c) $h(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}} = (x^2+x+1)^{-1/3}$

$= a(b(x))$

$a(x) = x^{-1/3}, \quad b(x) = x^2+x+1$

$h'(x) = a'(b(x)) \cdot b'(x)$

$= -\frac{1}{3} \cdot (x^2+x+1)^{-4/3} \cdot (2x+1)$ ☆

$= \dfrac{-(2x+1)}{3\cdot(x^2+x+1)^{4/3}}$

(d) $g(x^2)$ and $\sqrt[3]{g(x)}$

$g(x^2) = g(b(x))$ where $b(x) = x^2$

$\frac{d}{dx}(g(x^2)) = g'(b(x)) \cdot b'(x)$

$= g'(x^2) \cdot 2x$

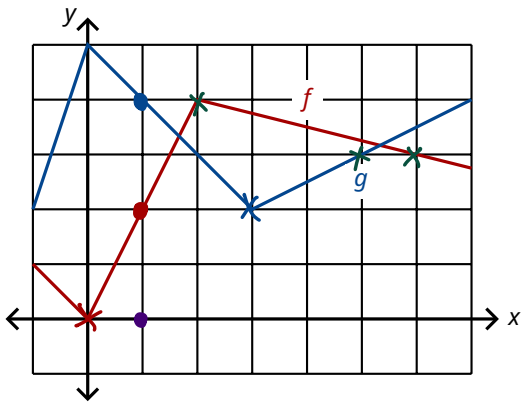
$\sqrt[3]{g(x)} = a(g(x))$ where $a(x) = x^{1/3}$

$a'(x) = \frac{1}{3}x^{-2/3}$

$\frac{d}{dx}(\sqrt[3]{g(x)}) = a'(g(x)) \cdot g'(x)$

$= \frac{1}{3} \cdot (g(x))^{-2/3} \cdot g'(x)$

Problem 2. The graphs of f and g are given below. Let $u(x) = f(g(x))$, $v(x) = g(f(x))$ and $w(x) = g(g(x))$.



(a) Compute $f(1)$, $g(1)$, $f'(1)$ and $g'(1)$.

$f(1) = 2 \rightsquigarrow f'(1) = 2-0/1-0 = 2$

$g(1) = 4 \rightsquigarrow g'(1) = 4-2/1-3 = -1$

(b) Compute $f'(g(1))$, $g'(f(1))$ and $g'(g(1))$.

$$f'(g(1)) = f'(4) = \frac{4-3}{2-6} = -\frac{1}{4}$$

$$g'(g(1)) = g'(4) = \frac{2-3}{3-5} = \frac{1}{2}$$

$g'(f(1)) = g'(2) = g'(1) = -1$ ($(1, g(1))$ & $(2, g(2))$ on the same line)

(c) Use (a) and (b) to compute $u'(1)$, $v'(1)$ and $w'(1)$ using the chain rule.

$u'(1) = f'(g(1)) \cdot g'(1)$

$= -\frac{1}{4} \cdot (-1)$

$= \frac{1}{4}$

$v'(1) = g'(f(1)) \cdot f'(1)$

$= -1 \cdot (2)$

$= -2$

$w'(1) = g'(g(1)) \cdot g'(1)$

$= \frac{1}{2} \cdot (-1)$

$= -\frac{1}{2}$

Example D. Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and is radius is $r(t)$.

(a) What do the derivatives $\frac{dV}{dr}$ and $\frac{dV}{dt}$ represent?

$\frac{dV}{dr}$ = instantaneous rate of change of Volume with respect to the radius

$\frac{dV}{dt}$ = instantaneous rate of change of Volume with respect to time

(b) Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$.

$V = \frac{4}{3}\pi r^3 \rightsquigarrow \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

Chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

Problem 3. *Please use Desmos for this Problem by visiting:* https://www.desmos.com/calculator/kaalwsaqlm

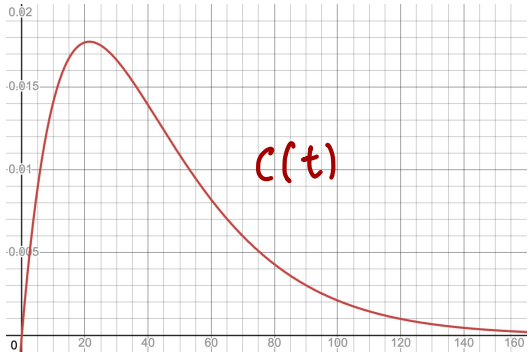
Blood alcohol content (BAC) after consumption of a single alcoholic drink (15mL ethanol) can be modeled by the concentration function

$$C(t) = 0.00225t \cdot e^{-0.0467t}$$

where t is the number of minutes after consumption of the drink and C is measured in g/dL. (dL = *decilitre* = 10^{-1} litre)

(a) Plot $C(t)$ on Desmos and discuss the features of the graph. Does this look like a reasonable model for BAC?

It starts at 0, has a delayed peak, then tapers off in an exponential way. Matches the intuition for how alcohol may enter and leave the blood stream



(b) What does $C'(t)$ represent? What would it mean to have $C'(t) = 0$?

$C'(t)$ represents the rate at which alcohol is being processed/removed from the bloodstream. $C'(t) = 0$ means instantaneous rate at $t = 0$. In this case, this is a peak of $C(t)$, where $C'(t)$ changes from positive to negative

(c) When does BAC reach its peak?

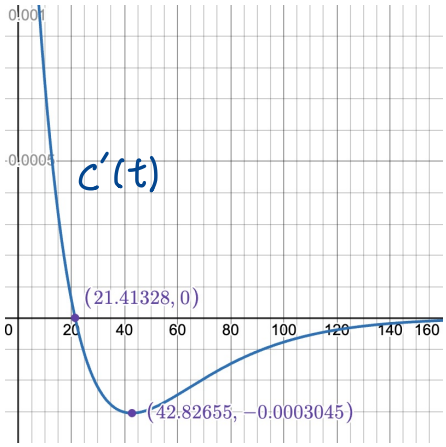
We may use Desmos to find t at which $C'(t) = 0$, which is $t \approx 21.4133$

At this peak, $C'(t) \approx 0.0177$

(d) When is BAC decreasing most rapidly?

Here we look for a t value where $C'(t)$ is most negative which would occur where $C''(t) = 0$

Using Desmos, we find $c'(t)$ is most negative when $t \approx 42.83$



Recap Problems

The aim is to review Day 8 material; so have the filled-in Day 8 notes in front of you as a reference!

Recap Problem 1. Compute the derivative of

$$\frac{2+\sqrt{x}}{1+5^x}$$

Recap Problem 2. Find the derivative of

$$\tan(x)\cdot\tan(x)+\frac{2\cos(x)}{1+\sin(x)}$$

Name: Deewang

Day 10. Implicit Differentiation

(based on Sections 3.5)

Learning Objectives

- Distinguish between explicitly and implicitly defined functions.
- Apply the Chain Rule to differentiate variables with respect to x.
- Find the equation of tangent lines to non-function curves (circles, hyperbolas).

Chain Rule. If f and g are differentiable functions such that we can consider their composition $f(g(x))$. Then,

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Implicit Differentiation

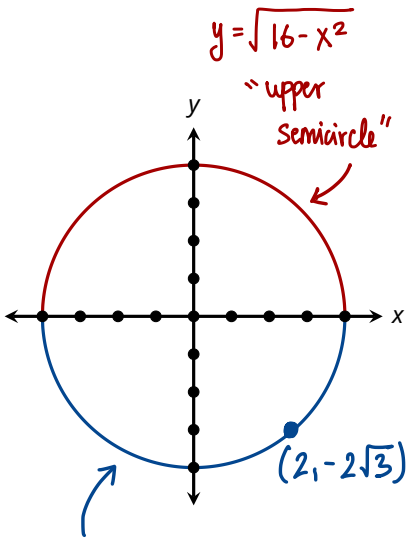
Example A (Motivating Example)**.** Suppose we want to find the tangent line to $x^2 + y^2 = 16$ at $(2, -2\sqrt{3})$.

We have the point; we need the slope! How can we find $\frac{dy}{dx}$?

One Way. $y^2 = 16 - x^2$

$y = \pm\sqrt{16 - x^2}$. Pick $y = -\sqrt{16 - x^2}$ since $(2, -2\sqrt{3})$ is on it

$-\sqrt{16 - x^2} = f(g(x))$ where $g(x) = 16 - x^2$, $f(x) = -\sqrt{x}$



$$\frac{dy}{dx} = \frac{-1}{2\sqrt{16 - x^2}} \cdot (-2x) = \frac{x}{\sqrt{16 - x^2}}$$

($f'(g(x))$, $g'(x)$)

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{2}{\sqrt{16 - 2^2}} = \frac{2}{\sqrt{12}} = \frac{2}{\sqrt{4 \cdot 3}} = \frac{2}{\sqrt{4}\cdot\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Another Way

Do not solve for y. Start differentiating $x^2 + y^2 = 16$

$$\frac{d}{dx}(16) = \frac{d}{dx}(x^2 + y^2)$$

$$\rightsquigarrow \quad 0 = \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x + \frac{d}{dx}(y^2)$$

$$0 = 2x + 2y \cdot \frac{dy}{dx} \quad \rightsquigarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\text{So, } \left.\frac{dy}{dx}\right|_{(2,-2\sqrt{3})} = \frac{-2}{-2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

y depends on x

y is like $f(x)$

y^2 is like $(f(x))^2$

$$\frac{d}{dx}\left((f(x))^2\right) = 2f(x) \cdot f'(x)$$

$$\frac{d}{dx}(y^2) = 2y \cdot y'$$

Equation of the tangent line:

$$y - (-2\sqrt{3}) = \frac{1}{\sqrt{3}}(x - 2)$$

Explicitly-defined Equation

Can you solve for y? Yes

$y = 8\sin(x)$

$xy = 8 \rightsquigarrow y = 8/x$

Implicitly-defined Equation

Can you solve for y? No!

$y^2 + xy + x^3 = 1$

Differentiating a Function in a Dependent Variable.

Let y be a dependent variable with respect to an independent variable x, therefore it makes sense to consider

$$y' = \frac{dy}{dx}$$

Suppose now you were given a function in y, say $g(y)$. Then, by the **chain rule**

$$\frac{d}{dx}(g(y)) = \frac{d}{dy}(g(y)) \cdot \frac{dy}{dx} = g'(y) \cdot y'$$

Example B. Consider the equation $x^2 - 3xy + x^2y^3 = 1$. Find $\frac{dy}{dx}$ at $(1, 0)$.

$$\frac{d}{dx}\left(x^2 - 3xy + x^2y^3\right) = \frac{d}{dx}(1)$$

product rule

$$(x^2)' - 3\left[(x)'y + x\cdot(y)'\right] + \left[(x^2)'\cdot y^3 + x^2\cdot(y^3)'\right] = 0$$

$$2x - 3(y + xy') + \left(2xy^3 + x^2\cdot(3y^2y')\right] = 0$$

— stop here. This is good!

For y' at $(1,0)$, plug in $x=1$ and $y=0$ and solve for y'

$$2\cdot 1 - 3(0 + 1\cdot y') + (2\cdot 1\cdot 0^3 + 1^2\cdot 3\cdot 0^2\cdot y') = 0$$

$$2 - 3y' = 0$$

$$\rightsquigarrow \quad \boxed{y' = 2/3}$$

Problem 1. Consider the hyperbola $x^2 - xy - y^2 = 1$

(a) Differentiate the equation $x^2 - xy - y^2 = 1$ implicitly. *Do not solve for y'.*

$$2x - \underbrace{(y + xy')}_{\text{product rule}} - 2yy' = 0$$

(b) Compute y' at the point $(2, 1)$ by plugging in $x = 2$ and $y = 1$ into your answer to (a).

$$2\cdot 2 - (1 + 2y') - 2\cdot 1\cdot y' = 0$$

$$4 - (1 + 2y') - 2y' = 0 \quad \rightsquigarrow \quad \boxed{y' = 3/4}$$

(c) Find the equation of the tangent line at the point $(2, 1)$.

$$y - 1 = \frac{3}{4}(x - 2)$$

(d) If there is time, use Desmos to plot this hyperbola together with the tangent line you found.

Example C. Implicitly differentiate $1 + \sin(xy) = e^y - x$ and compute y' at $(0, 0)$.

$$\frac{d}{dx}(1 + \sin(xy)) = \frac{d}{dx}(e^y - x)$$

$$0 + \underbrace{\cos(xy) \cdot (xy)'}_{\text{chain rule}} = e^y \cdot y' - 1$$

$$\cos(xy) \cdot (y + xy') = e^y \cdot y' - 1$$

$$\cos(0 \cdot 0) \cdot (0 + 0 \cdot y') = e^0 \cdot y' - 1$$

$$0 = y' - 1$$

$$\rightsquigarrow \boxed{y' = 1}$$

Problem 2. Consider the curve: $\cos(x + y) = 2 - e^y$

(a) Differentiate the equation $\cos(x + y) = 2 - e^y$ implicitly. *Do not solve for y'.*

$$-\sin(x+y) \cdot (x+y)' = 0 - e^y \cdot y'$$

$$-\sin(x+y) \cdot (1+y') = -e^y \cdot y'$$

(b) Compute y' at the point $(0, 0)$.

$$-\sin(0+0)\cdot(1+y') = -e^0\cdot y'$$

$\sin(0)=0$

$$0 = -y' \quad \rightsquigarrow \quad \boxed{y'=0}$$

(c) Find the equation of the tangent line at the point $(0, 0)$.

$$y - 0 = 0\cdot(x-0)$$

(d) If there is time, use Desmos to plot this hyperbola together with the tangent line you found.

Example D. Consider $1 + \sin(xy) = e^y - x$. Compute y'' at $(0, 0)$.

When we differentiated the first time (Example C), we got:

$$\cos(xy)\cdot(y+xy') = e^y\cdot y' - 1$$

We differentiate again:

$$-\sin(xy)\cdot(y+xy')\cdot(y+xy') + \cos(xy)\cdot(y'+y'+xy'') = e^y\cdot y'\cdot y' + e^y\cdot y''$$

To find y'' at $(0,0)$, plug in $(x,y)=(0,0)$ & y' at $(0,0)$ $= 1$ (Example C)

$$-\sin 0\cdot(0+0\cdot 1)\cdot(0+0\cdot 1) + \cos(0)\cdot(1+1+0\cdot y'') = e^0\cdot 1\cdot 1 + e^0\cdot y''$$

$$2 = 1 + y'' \quad \rightsquigarrow \quad \boxed{y''=1}$$

Problem 3. Consider $\cos(x + y) = 2 - e^y$ from Problem 2. Compute y'' at $(0, 0)$.

When we differentiated the first time (Problem 2), we got:

$$-\sin(x+y)\cdot(1+y') = -e^y\cdot y'$$

We differentiate again:

$$-\cos(x+y)\cdot(1+y')\cdot(1+y') - \sin(x+y)\cdot y'' = -e^y\cdot y'\cdot y' - e^y\cdot y''$$

To find y'' at $(0,0)$, plug in $(x,y) = (0,0)$ & y' at $(0,0)$ $= 0$ (Problem 2)

$$-\cos(0+0)\cdot(1+0)\cdot(1+0) - \sin(0+0)\cdot y'' = -e^0\cdot 0\cdot 0 - e^0\cdot y''$$

$$-1 = -y'' \rightsquigarrow \boxed{y'' = 1}$$

Differentiating the Logarithm

Using Implicit Differentiation to Compute the Derivative of the Function $y = \ln(x)$

$y = \ln(x)$

What is y'?

$e^y = x$ ~differentiating~> $e^y\cdot y' = 1$

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

Derivative of the Natural Logarithm. Given the function $\ln(x)$, we have

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Recap Problems

The aim is to review Day 9 material; so have the filled-in Day 9 notes in front of you as a reference!

Recap Problem 1. Compute the derivative of

$$g(\theta) = 2^{3\sin(\theta)}$$

Recap Problem 2. Compute the derivative of

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

This requires both quotient and chain rule.

Name: Deewang

Day 13. Logarithmic Differentiation

(based on Sections 3.2, 3.3)

Learning Objectives

- Differentiate natural and general logarithmic functions.
- Use logarithmic differentiation for "variable-to-the-variable" powers.
- Simplify differentiation of large products/quotients using log laws.

Derivative of Logarithms

Using Implicit Differentiation to Compute the Derivative of the Function $y = \ln(x)$

$y = \ln(x) \rightsquigarrow e^y = x \xrightarrow{\text{diff. implicitly}} e^y \cdot y' = 1 \rightsquigarrow y' = \frac{1}{e^y} = \frac{1}{x}$

$y' = ?$

Derivative of the Natural Logarithm. Given the function $\ln(x)$, we have

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Example A. Let $a > 0$, then

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Let us compute $\frac{d}{dx}(\log_a(x))$.

$$\frac{d}{dx}\left(\frac{\ln(x)}{\ln(a)}\right) = \frac{d}{dx}\left(\frac{1}{\ln(a)} \cdot \ln(x)\right) = \frac{1}{\ln(a)} \frac{d}{dx}(\ln(x))$$

$$= \frac{1}{\ln(a)} \cdot \frac{1}{x}$$

Derivative of the General Logarithm Function. Given the function $\log_a(x)$ where $a > 0$, we have

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$$

Problem 1. Compute the derivatives of:

(a) $\ln(t^2+1)$

chain rule

outside : $\ln(t)$; inside : t^2+1

$$\frac{d}{dt}\left(\ln(t^2+1)\right) = \frac{1}{t^2+1}\cdot 2t$$

(b) $x\cdot\ln(x)$ $\quad (x\cdot\ln(x))'$

$$= (x)'\cdot\ln(x) + x\cdot(\ln(x))'$$

$$= \ln(x) + x\cdot\frac{1}{x}$$

$$= \ln(x) + 1$$

(c) $-\ln(\cos(x))$

$$\left(-\ln(\cos(x))\right)' = -\frac{1}{\cos(x)}\cdot(-\sin(x))$$

$$= \frac{\sin(x)}{\cos(x)} = \tan(x)$$

(d) $\ln(\ln(x))$

$$\left(\ln(\ln(x))\right)' = \frac{1}{\ln(x)}\cdot\frac{1}{x}$$

Logarithmic Differentiation

We know how to differentiate

power functions, for example:

variable → x^2 ← constant

$$(x^2)' = 2x$$

exponential functions, for example:

constant → 2^x ← variable

$$(2^x)' = 2^x\cdot\ln(2)$$

What about the derivative of functions like: x^x ← variable

More generally, functions of the form:

$$f(x)^{g(x)}$$

Method of Logarithmic Differentiation. This is a tool that combines the algebra of logarithms with implicit differentiation to find derivatives of expressions that may otherwise not seem to fit with other rules of differentiation.

Steps in Logarithmic Differentiation

1. Take natural logarithms on both sides of $y = f(x)$ and use the Logarithm Rules to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

Recalling Logarithm Rules: Let $a > 0$ and b and c can be anything:

- $\log_a(bc) =$ $\log_a(b) + \log_a(c)$
- $\log_a\left(\frac{b}{c}\right) =$ $\log_a(b) - \log_a(c)$
- $\log_a(b^r) =$ $r \cdot \log_a(b)$

Warning: The following "rules" are WRONG:

- $\log_a(b + c) =$ $\log_a(b) + \log_a(c)$ is WRONG
- $\log_a(b - c) =$ $\log_a(b) - \log_a(c)$ is WRONG
- $\log_a(bc) =$ $\log_a(b) \cdot \log_a(c)$ is WRONG
- $\log_a\left(\frac{b}{c}\right) =$ $\log_a(b) / \log_a(c)$ is WRONG

Example B. Let us compute $\frac{d}{dx}(x^x)$

Step 0. $y = x^x$

Step 1. $\ln(y) = \ln(x^x)$

$= x \cdot \ln(x)$

Step 2.

$\frac{1}{y} \cdot y' = \ln(x) + x \cdot \frac{1}{x}$

Step 3. $y' = (\ln(x) + 1) \cdot y$

$= (\ln(x) + 1) \cdot x^x$

Example C. Let us compute the derivative of $(x^2+x)^{\sin(x)}$

Step 0. $y = (x^2+x)^{\sin(x)}$

Step 1. $\ln(y) = \ln\left((x^2+x)^{\sin(x)}\right) = \sin(x)\cdot\ln(x^2+x)$

Step 2. $\frac{1}{y}\cdot y' = \cos(x)\cdot\ln(x^2+x) + \sin(x)\cdot\frac{1}{x^2+x}\cdot(2x+1)$

Step 3. y'

$$= (x^2+x)^{\sin(x)}\left(\cos(x)\cdot\ln(x^2+x) + \sin(x)\cdot\frac{2x+1}{x^2+x}\right)$$

Problem 2. Find $\frac{dy}{dx}$ for the following functions:

(a) $y = (x^2+7)^x \rightsquigarrow \ln(y) = \ln\left((x^2+7)^x\right) = x\cdot\ln(x^2+7)$

$$\frac{1}{y}\cdot y' = \ln(x^2+7) + x\cdot\frac{1}{x^2+7}\cdot 2x$$

$$y' = y\cdot\left(\ln(x^2+7) + \frac{2x^2}{x^2+7}\right)$$

$$= (x^2+7)^x\cdot\left(\ln(x^2+7) + \frac{2x^2}{x^2+7}\right)$$

(b) $y = (x \cdot \ln x)^{\cos(x)}$ $\rightsquigarrow$ $\ln(y) = \cos(x) \cdot \ln(x \cdot \ln(x))$

$$\frac{1}{y} \cdot y' = -\sin(x) \cdot \ln(x \cdot \ln(x)) + \cos(x) \cdot \frac{1}{x \cdot \ln(x)} \cdot (\ln(x) + 1)$$

$$y' = y \cdot \left(-\sin(x) \cdot \ln(x \cdot \ln(x)) + \cos(x) \cdot \frac{1}{x \cdot \ln(x)} \cdot (\ln(x) + 1) \right)$$

$$y' =$$

$$(x \cdot \ln(x))^{\cos(x)} \cdot \left(-\sin(x) \cdot \ln(x \cdot \ln(x)) + \cos(x) \cdot \frac{1}{x \cdot \ln(x)} \cdot (\ln(x) + 1) \right)$$

(c) $y = (\ln x)^{\ln x}$

$$\ln(y) = \ln(x) \cdot \ln(\ln(x))$$

$$\frac{1}{y} \cdot y' = \frac{1}{x} \cdot \ln(\ln(x)) + \ln(x) \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}$$

$$y' = y \cdot \left(\frac{\ln(\ln(x))}{x} + \frac{1}{x} \right)$$

$$= (\ln(x))^{\ln(x)} \cdot \left(\frac{\ln(\ln(x))}{x} + \frac{1}{x} \right)$$

Recap Problems

The aim is to review Day 10 material; so have the filled-in Day 10 notes in front of you as a reference!

Recap Problem 1. For the equation $x^3 - xy + xy^2 = 1$, compute y' at $(1, 0)$.

Recall. Differentiate but do not solve for y'. Plug in your point value at then solve for y' to obtain the answer.

Recap Problem 2. For the equation $e^{x+y} = \cos(xy)$, compute y' at $(0, 0)$. (useful: $\sin(0) = 0$ and $\cos(0) = 1$)

Recall. Differentiate but do not solve for y'. Plug in your point value at then solve for y' to obtain the answer.

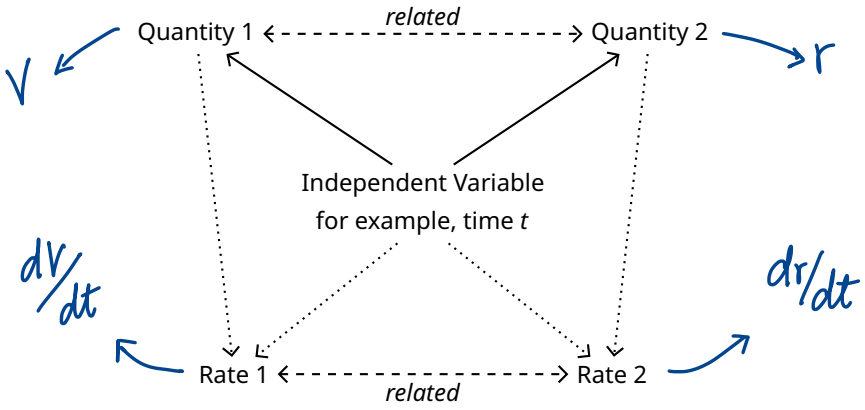
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Day 14. Related Rates

(based on Sections 3.9)

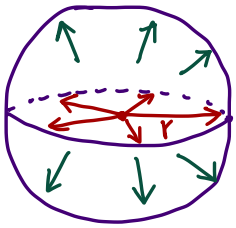
Learning Objectives

- Identify the known and unknown rates in a word problem.
- Construct an equation that relates the quantities involved.
- Use implicit differentiation with respect to time to solve for an unknown rate.



Example A. Air is being pumped into a spherical balloon so that the volume of the balloon is increasing at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Movie



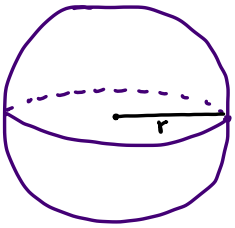
Volume V changing with time t

Radius r changing with time t

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

Snapshot



$2r$ = diameter = 50 cm

$r = 25$cm

$\frac{dV}{dt} = 100\ cm^3/s$ $\quad \frac{dr}{dt} = ?$

$\rightarrow 100 = 4\pi \cdot (25)^2 \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{100}{4\pi \cdot (25)^2} = \frac{1}{25\pi} \text{ cm/s}$$

unit of r

unit of t

Problem Solving Strategy for Related Rates.

(1) *Read* the problem carefully. Note that the problem involves *two rates*, one given and the other unknown. Key phrases include *how fast* and *at a/what rate*.

(2) *Draw* a picture. Introduce *notation* - assign symbols to all quantities that are functions of time.

(3) *Identify* the givens and unknowns. Attend to *units*.

(4) Find an *equation relating* the two variables associated with the rates. If necessary, use geometry to eliminate additional variables through substitution.

(5) Use the Chain Rule to *differentiate implicitly* both sides of the equation *with respect to* time.

(6) *Solve* for the unknown rate.

(7) *Substitute* in the given information.

Warning. *Do no substitute in the given information too early*. Differentiate first!

Example B. A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water increasing? The volume of a cylinder is $\pi r^2 h$, where r is the radius and h is the height.

Volume V changing, units = m^3

Height h changing, units = m

Radius fixed r = 5m

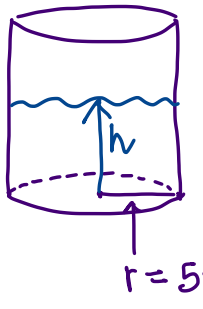
$V = \pi r^2 h = \pi \cdot 5^2 \cdot h = 25\pi h$

$$\frac{dV}{dt} = 25\pi \cdot \frac{dh}{dt}$$

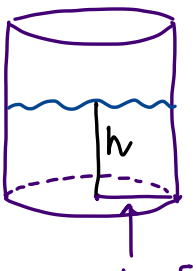
$\frac{dV}{dt} = 3\ m^3/min$ $\quad \frac{dh}{dt} = ?$

$3 = 25\pi \cdot \frac{dh}{dt} \rightsquigarrow \frac{dh}{dt} = \frac{3}{25\pi}$ m/min

Movie



Snapshot



Problem 1. The radius of a sphere is increasing at a rate of 2 cm/min. How fast is the surface area of the ball increasing when the diameter is 8 cm? The surface area of a sphere is $4\pi r^2$, where r is the radius.

Surface area S changing, units $= cm^2$

$S = 4\pi r^2$

Radius r changing, units $= cm$

$$\frac{dS}{dt} = 4\pi \cdot 2r \frac{dr}{dt} = 8\pi r \frac{dr}{dt}$$

Snapshot: $2r =$ diameter $= 8\,cm$, $\frac{dr}{dt} = 2\,cm/min$, $\frac{dS}{dt} = ?$

$\rightsquigarrow r = 4\,cm$

$$\frac{dS}{dt} = 8 \cdot \pi \cdot \left(\frac{8}{2}\right) \cdot 2 = 64\pi\ cm^2/min$$

Problem 2. A large block of ice in the shape of a perfect cube is melting such that it maintains its cubic shape. The volume of the ice is decreasing at a constant rate of 27 cm^3/min. How fast is the length of the side of the cube decreasing when the side length is exactly 3 cm? The volume of a cube is s^3, where s is the length of the side.

Volume V of cube changing, units $= cm^3$

$V = s^3$

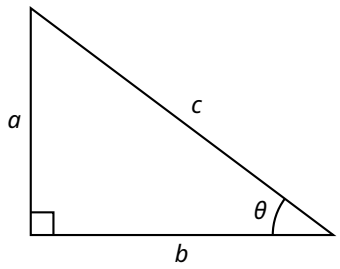
Side length s of cube changing, units $= cm$

$$dV/dt = 3s^2 \cdot ds/dt$$

Snapshot: $s = 3\,cm$, $dV/dt = -27\,cm^3/min$ (decreasing), $ds/dt = ?$

$$-27 = 3 \cdot 3^2 \frac{ds}{dt} \rightsquigarrow ds/dt = \frac{-27}{3 \cdot 3^2} = -1\ cm/min$$

Right-angled Triangles & Trigonometric Functions. For the next few problems, it will be useful to recall following facts about right-angled triangles. Let θ be one non-right-angle angles.



Pythagorean Theorem

$$a^2 + b^2 = c^2$$

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}$$

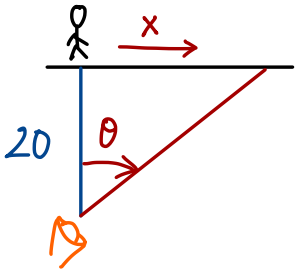
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}$$

$$\tan(\theta) = \frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$$

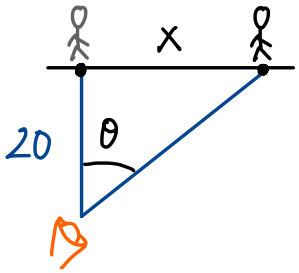
Example C. A person walks across a stage at a speed of 4 ft/s. A spotlight shines on this person from a location 20 feet from the person's path, rotating to continue shining on the person as they walk across the stage. At what rate is the spotlight rotating when the person is 15 ft from the closest point on the path to the spotlight?

(a) Draw a picture and introduce notation for the quantities discussed in the problem statement.

Movie



Snapshot



x is the distance from the person to the closest point on the path to the spotlight (in ft)

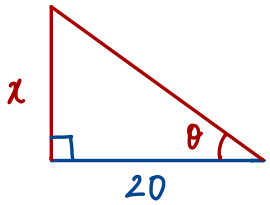
θ is the angle between the beam of light and the perpendicular to the path (in radians)

(b) Use this notation to state what is known, and what is unknown in the problem.

$\frac{dx}{dt} = 4$ ft/s $\quad \frac{d\theta}{dt} = ?$ when $x = 15$ ft

(c) Provide an equation that relates the known and unknown quantities.

$$\tan(\theta) = x/20$$



(d) Differentiate the equation from (c).

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{20} \frac{dx}{dt}$$

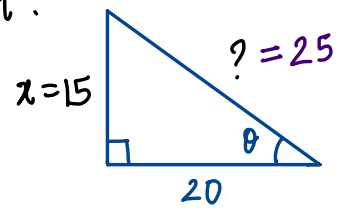
(e) Substitute known information into the result from (d) and solve for the unknown rate.

$$\frac{1}{\cos^2(\theta)} \frac{d\theta}{dt} = \frac{1}{20} \cdot 4 = \frac{1}{5}$$

$$\frac{d\theta}{dt} = \frac{1}{5} \cdot \cos^2(\theta)$$

$$= \frac{1}{5} \cdot \left(\frac{4}{5}\right)^2 = \frac{16}{125} \text{ rad/s}$$

Snapshot :



$$?^2 + 15^2 = 20^2$$

$$\rightsquigarrow ? = 25$$

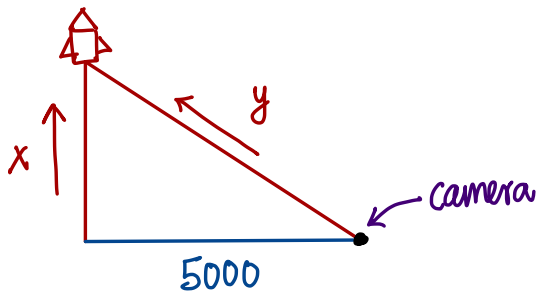
$$\cos\theta = \frac{20}{25} = \frac{4}{5}$$

Problem 3. A rocket is launched vertically and is tracked by a camera positioned on the ground 5000 feet from the launchpad. The rocket rises at a constant speed of 600 ft/s. How fast is the distance between the camera and the rocket changing when the rocket is 12000 feet above the ground?

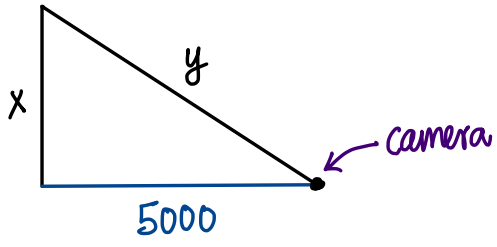
Do not rush to solve the problem: I want clear answers for each step/part.

(a) Draw a picture and introduce notation for the quantities discussed in the problem statement.

Movie



Snapshot



(b) Use this notation to state what is known, and what is unknown in the problem.

$\frac{dx}{dt} = 600$ ft/s $\qquad \frac{dy}{dt} = ?$ when $x = 12000$ ft

(c) Provide an equation that relates the known and unknown quantities.

$$x^2 + 5000^2 = y^2$$

(d) Differentiate the equation from (c).

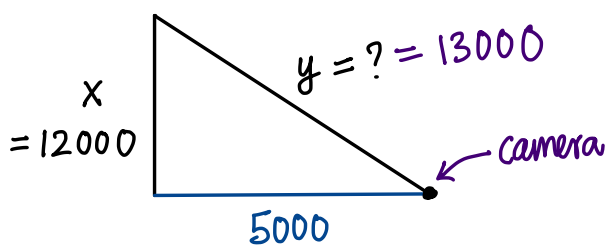
$$2x \cdot \frac{dx}{dt} = 2y \cdot \frac{dy}{dt}$$

(e) Substitute known information into the result from (d) and solve for the unknown rate.

$$2 \cdot 12000 \cdot 600 = 2y \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{2 \cdot 12000}{2 \cdot y} \cdot 600$$

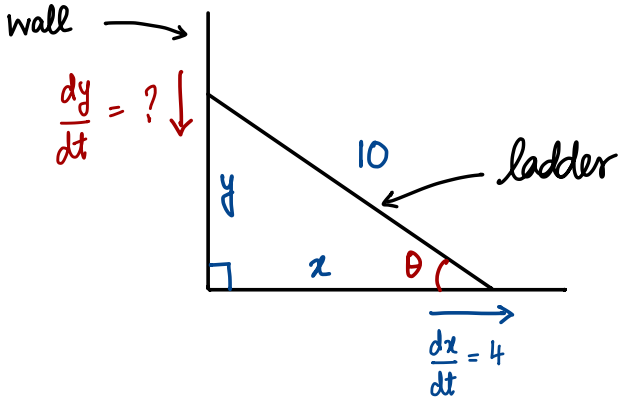
$$= \frac{12000}{13000} \cdot 600 = \frac{7200}{13} \text{ ft/s}$$



$$12000^2 + 5000^2 = ?^2$$

$$?^2 = 13000$$

Problem 4. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 4 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6ft from the wall? How fast is the angle between the ladder and the ground changing at this same moment?



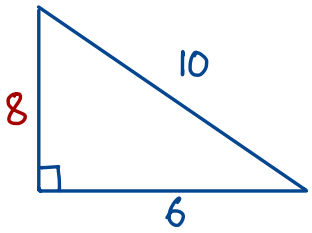
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Given $\frac{dx}{dt} = 4$, we wish to compute $\frac{dy}{dt}$ when $x = 6$ ft

(1) Using $x^2 + y^2 = 10^2$ we differentiate with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \rightsquigarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$, we have

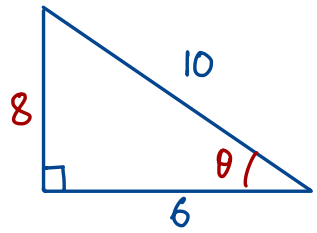


$$\frac{dy}{dt} = -\frac{6}{8} \cdot 4 = -3 \text{ ft/s}$$

(2) Using $\cos\theta = \frac{x}{10}$ we differentiate with respect to t

$$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dx}{dt} \quad \rightsquigarrow \quad \frac{d\theta}{dt} = -\frac{1}{\sin\theta} \cdot \frac{1}{10} \cdot \frac{dx}{dt}$$

When $x = 6$, we have



$$\frac{d\theta}{dt} = -\frac{10}{8} \cdot \frac{1}{10} \cdot 4 = \frac{-1}{2} \text{ rad/s}$$

Recap Problems

The aim is to review Day 13 material; so have the filled-in Day 13 notes in front of you as a reference!

Recap Problem 1. Find the derivative of $\sqrt{\ln(x^4)}$.

Recap Problem 2. Compute the derivative of

$$y = x^2 \cdot x^{\cos(x)}$$

using the method of logarithmic differentiation.

Name: _____

Day 15. Related Rates & Error Approximation

(based on Sections 3.9, 3.10)

Learning Objectives

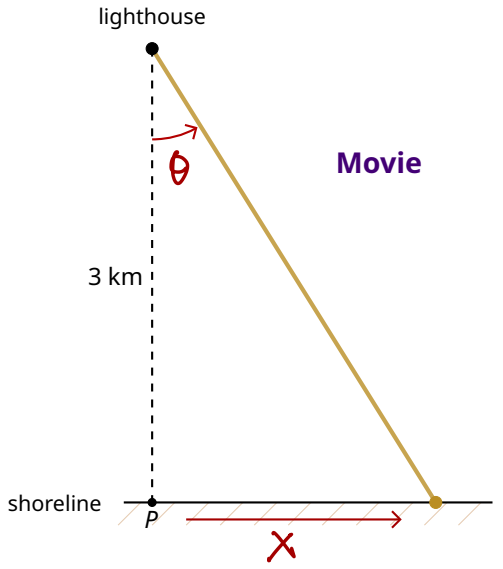
- Solve Related Rates problems involving changing radius, angles, lengths.
- Define differentials and understand their relationship to actual change.
- Estimate relative and percentage error in physical measurements.

Problem 1. A lighthouse sits 3 km off the coast of a straight shoreline. The point on the shore closest to the lighthouse is labeled P. The lighthouse beam rotates at a constant rate of 4 revolutions per minute. Determine the speed (in km/min) at which the beam of light travels along the shoreline at the moment the beam is 1 km away from point P.

Make sure you have converted angle measurements into radians: ***1 revolution* = 360° = 2π *radians***

I have drawn the "movie" picture.

Start by assigning variables to things that are changing with time.



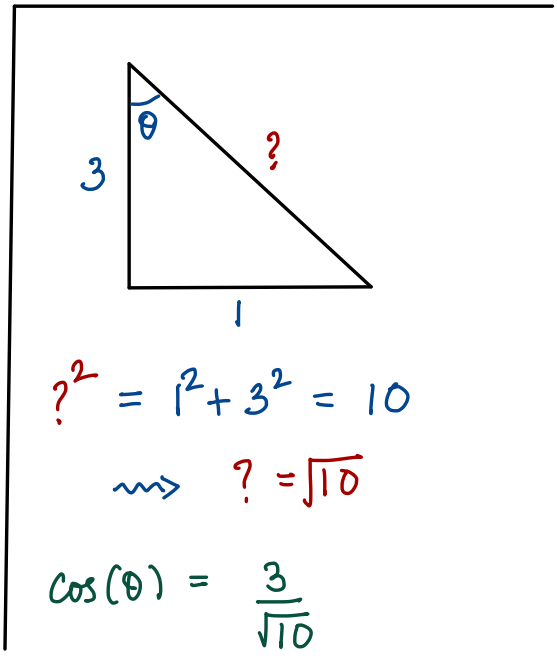
$\frac{d\theta}{dt} = 4 \text{ rev/min} = 4 \cdot 2\pi \text{ rad/min}$

$= 8\pi \text{ rad/min}$

$\frac{dx}{dt} = ?$ when $x = 1$ km

$\tan\theta = \frac{x}{3} \rightsquigarrow \sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{3} \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt} = \frac{3}{\cos^2(\theta)} \cdot \frac{d\theta}{dt}$



$?^2 = 1^2 + 3^2 = 10$

$\rightsquigarrow ? = \sqrt{10}$

$\cos(\theta) = \frac{3}{\sqrt{10}}$

$\frac{dx}{dt} = \frac{3}{(3/\sqrt{10})^2} \cdot 8\pi = \frac{10}{9} \cdot 3 \cdot 8\pi$

$= \frac{80\pi}{3}$ km/min

Problem 2. The volume of a spherical ball is increasing at a rate of 4 cm^3/min. At what rate is the surface area of the ball increasing when the radius is 8 cm?

1. First express dV/dt and dS/dt in terms of dr/dt.
2. Use the information given to compute dr/dt.
3. Use dr/dt to compute dS/dt.

Volume $V = \frac{4}{3}\pi r^3$; Surface Area $S = 4\pi r^2$ where r is the radius

(1) $V = \frac{4}{3}\pi r^3 \rightsquigarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$, $S = 4\pi r^2 \rightsquigarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = 4\ cm^3/min$ $\frac{dS}{dt} = ?$ when $r = 8\ cm$

(2) $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \rightsquigarrow 4 = 4\pi \cdot (8)^2 \cdot \frac{dr}{dt} \rightsquigarrow \frac{dr}{dt} = \frac{1}{64\pi}$

(3) $\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} \rightsquigarrow \frac{dS}{dt} = 8\pi \cdot (8) \cdot \frac{1}{64\pi} = 1\ cm^2/min$

Error Approximation

Approximating Error. Let $y = f(x)$ be a function that depends on x.

Let Δx be some small number, and we may consider x and its change with Δx: $x + \Delta x$

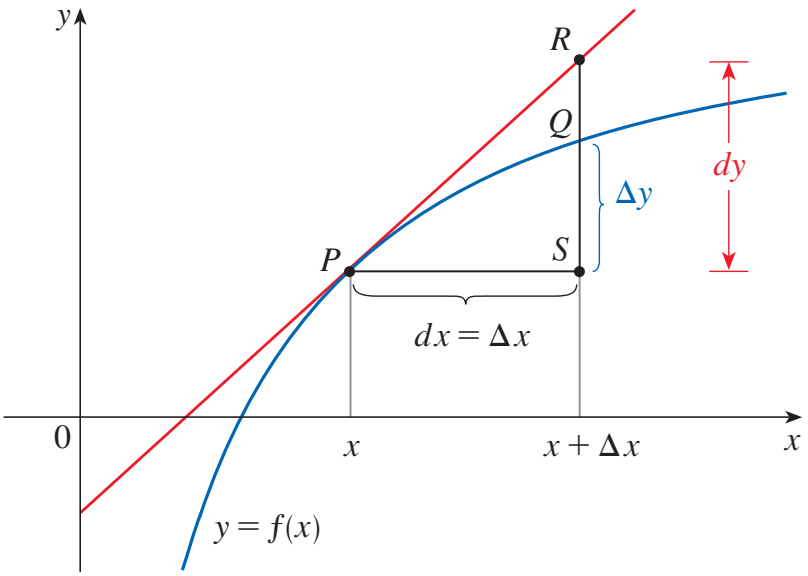
With these, we may consider the corresponding:

$$f(x) \quad \text{and} \quad f(x + \Delta x)$$

We are then interested in computing or estimating:

$$\Delta y = f(x + \Delta x) - f(x)$$

The more complicated $y = f(x)$ is the harder it is to compute this numerically.



Differentials. Let $y = f(x)$ be a differentiable function. The **differential** dx is an *independent variable*; that is, dx can be given the value of any real number.

The **differential** dy is a *dependent variable* defined in terms of dx using the equation

$$dy = f'(x)\,dx$$

Usually, $dx = \Delta x$ but Δy and dy are different

$$\Delta y = f(x+\Delta x) - f(x) \approx dy = f'(x)\,dx \quad (\to = \Delta x)$$

Smaller the Δx, the better dy approximates Δy. This is evident in the image above.

Relative error of y is defined to be

$$\frac{dy}{y} \approx \frac{\Delta y}{y}$$

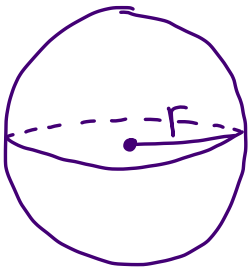
Percentage error of y is defined to be

$$\frac{dy}{y} \times 100$$

Main Idea: $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{"cancelling } dt\text{"} \quad dV = 4\pi r^2\,dr$$

Example A. The circumference of a sphere was measured to be 8 cm with a possible error of 0.5 cm.



Circumference $C = 2\pi r$

$8 = 2\pi r \rightsquigarrow r = 4/\pi$ cm

$dC = 2\pi\, dr$ and $dC = 0.5$ cm

$dr = dC/2\pi = 0.5/2\pi$ cm

(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?

$S = 4\pi r^2 = 4\pi \cdot 4/\pi \cdot 4/\pi = 64/\pi$ cm^2

$dS = 8\pi r\, dr = 8\pi \cdot 4/\pi \cdot 0.5/2\pi = 8/\pi$ cm^2

$\frac{dS}{S} = \frac{8/\pi}{64/\pi} = \frac{1}{8} \approx 0.125$

12.5%.

(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

$V = 4/3\,\pi r^3 = 4/3\,\pi \cdot (4/\pi)^3 = 256/3\pi^2$ cm^3

$dV = 4\pi r^2\, dr = 4\pi \cdot (4/\pi)^2\, 0.5/2\pi = 16/\pi^2$ cm^3

$dV/V = (16/\pi^2) \Big/ (256/3\pi^2) = 3/16 \approx 0.1875$

18.75%.

Problem 3. The edge of a cube is measured to be 30 cm with a possible error in the measurement of 0.1 cm. Estimate the maximum possible error, relative error and percentage error in computing

(a) the volume of the cube; and (b) the surface area of the cube

(1) Let s be the length of the edge of the cube; volume $V = s^3$ and surface area $A = 6s^2$.
Compute dV and dA in terms of ds. They approximate ΔV and ΔA.

$V = s^3 \rightsquigarrow dV = 3s^2\,ds$ $\quad \Delta V \approx 3s^2\,\Delta s$

$A = 6s^2 \rightsquigarrow dA = 12s\,ds$ $\quad \Delta A \approx 12s\,\Delta s$

(2) Use $s = 30$ and $ds = 0.1$ to compute maximal possible error $\Delta V \approx dV$ and $\Delta A \approx dA$, using (1).

$dV = 3s^2\,ds = 3(30)^2 \cdot (0.1) = 270\ \text{cm}^3$ $\quad \Delta V \approx 270\ \text{cm}^3$

$dA = 12s\,ds = 12(30) \cdot (0.1) = 36\ \text{cm}^2$ $\quad \Delta A \approx 36\ \text{cm}^2$

(3) Relative error is given by $\frac{\Delta V}{V} \approx \frac{dV}{V}$ and $\frac{\Delta A}{A} \approx \frac{dA}{A}$. Compute this for $s = 30$ and $ds = 0.1$.

$\frac{dV}{V} = \frac{270}{(30)^3} = 0.01$ $\quad \frac{\Delta V}{V} \approx 0.01$

$\frac{dA}{A} = \frac{36}{6 \cdot (30)^2} = 0.0067$ $\quad \frac{\Delta A}{A} \approx 0.0067$

(4) Percentage error is $100 \times$ (relative error). Use (3) to answer this.

Percentage error of volume $\approx \frac{dV}{V} \times 100 = 1\%$

Percentage error of area $\approx \frac{dA}{A} \times 100 = 0.67\%$

Recap Problems

The aim is to review Day 14 material; so have the filled-in Day 14 notes in front of you as a reference!

Recap Problem 1. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

recall that distance travelled = speed $\times$ time

Name: Deewang

Day 16. The Area Problem & the Definite Integral

(based on Sections 5.1, 5.2)

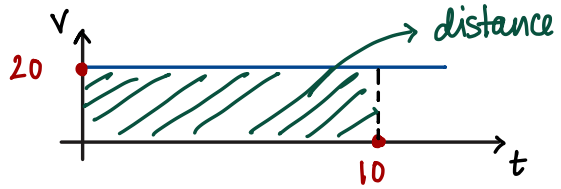
Learning Objectives

- Relate the area and distance problems.
- Approximate the area under the graph of a function using right, left and midpoints.
- Understand the Definite Integral as both the limit of a Riemann Sum and as signed area.

The Distance Problem. Suppose an object travels for 10 seconds with a constant velocity of 20 m/s. What is the distance that this object travelled?

distance = velocity x time = 20 m/s x 10 s = 200 m

This is equivalent to the following:

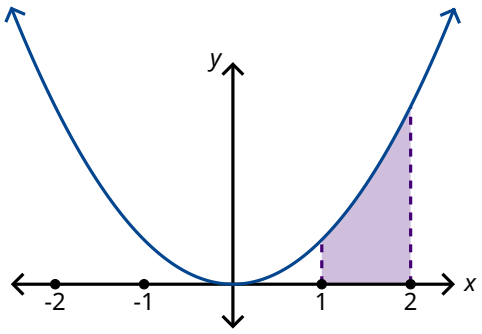


area under the velocity curve

Therefore, in general,

distance is area under the velocity graph!

The Area Problem. Let us consider function $f(x) = x^2$ which has the following graph:

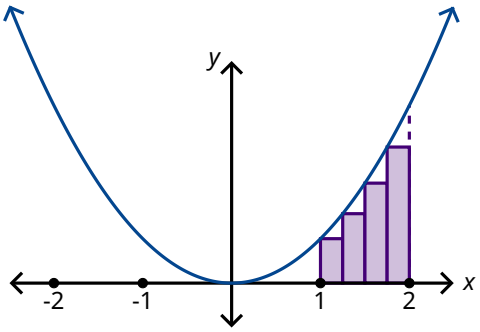


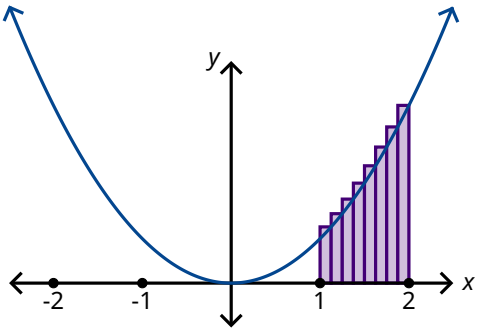
and suppose we wish to compute the area under the curve between $x = 1$ and $x = 2$ (i.e., area of the shaded region).

We do not have a nice formula to compute the area under the curve since this is *not* a familiar shape like a rectangle or a triangle or a circle.

The main idea is to approximate with shapes using areas we do know! For our discussions, we will use

rectangles!

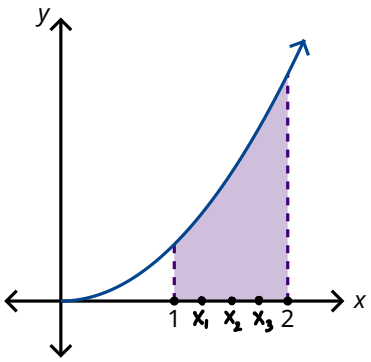




Step 1. Look at the interval $[a, b]$ between which you wish to compute the area under the curve $y = f(x)$. For us,

$a =$ 1

$b =$ 2



Step 2. Decide how many rectangles we will use. Call it n. For us, let us choose $n =$ 4

Step 3. Compute the width of each rectangle, which is given by $\Delta x = \dfrac{b-a}{n}$. For us,

$$\Delta x = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

Step 4. These rectangles will subdivide the interval $[a, b]$ into n subintervals

$$[x_0, x_1],\ [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of these subintervals are given by

$$x_1 = a + \Delta x \qquad x_2 = a + 2\Delta x \qquad x_3 = a + 3\Delta x \qquad \cdots \qquad x_i = a + i \cdot \Delta x \qquad \cdots$$

For us, $x_0 = 1$, $x_4 = 2$

$x_1 = 1 + 0.25 = 1.25$, $x_2 = 1 + 2 \cdot 0.25 = 1.5$, $x_3 = 1 + 3 \cdot 0.25 = 1.75$

giving us the subintervals:

$[x_0, x_1]$ $[x_1, x_2]$ $[x_2, x_3]$ $[x_3, x_4]$

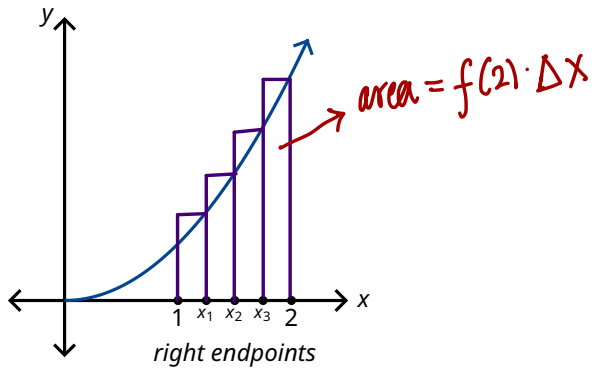
$[1, 1.25]$ $[1.25, 1.5]$ $[1.5, 1.75]$ $[1.75, 2]$

Step 5. We have two choices for heights of our rectangles for each subinterval.

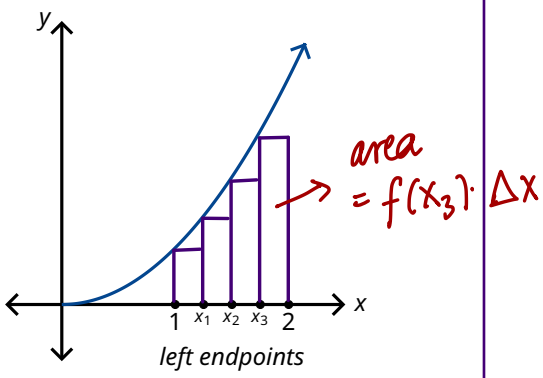
We may take as height the function outputs of *right endpoints*: $f(x_1),\ f(x_2),\ f(x_3),\ \ldots,\ f(x_n)$; or

We may take as height the function outputs of *left endpoints*: $f(x_0),\ f(x_1),\ f(x_2),\ \ldots,\ f(x_{n-1})$.

For us, it looks like the following:



right endpoints



left endpoints

Step 6. We may then approximate area

using right endpoints

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

using left endpoints

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

For us,

$$R_4 = f(1.25)\,0.25 + f(1.5)\,0.25 + f(1.75)\,0.25 + f(2)\,0.25$$

$$= (1.25)^2 0.25 + (1.5)^2 0.25 + (1.75)^2 0.25 + (2)^2 0.25$$

$$= 2.71875$$

$$L_4 = f(1)\,0.25 + f(1.25)\,0.25 + f(1.5)\,0.25 + f(1.75)\,0.25$$

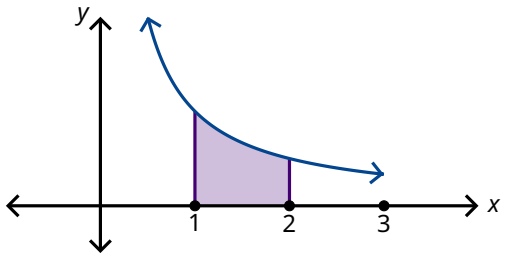
$$= (1)^2 0.25 + (1.25)^2 0.25 + (1.5)^2 0.25 + (1.75)^2 0.25$$

$$= 1.96875$$

Step 7. Finally, we obtain the true area by letting n go to infinity (i.e., making our approximations finer)

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

Problem 1. Consider the function $f(x) = 1/x$ and consider the area under the graph between $x = 1$ to $x = 2$.



(a) Use four approximating rectangles and right endpoints to estimate the area under the graph between these points. Add these rectangles to your graph. Is your estimate an overestimate or underestimate?

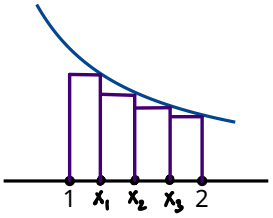
$a = 1, \quad b = 2, \quad n = 4, \quad \Delta x = \frac{b-a}{n} = \frac{1}{4} = 0.25$

$x_0 = 1, \quad x_4 = 2, \quad x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75$

$R_4 = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\Delta x$

$= \frac{1}{1.25}\cdot 0.25 + \frac{1}{1.5}\cdot 0.25 + \frac{1}{1.75}\cdot 0.25 + \frac{1}{2}\cdot 0.25$

≈ 0.6345



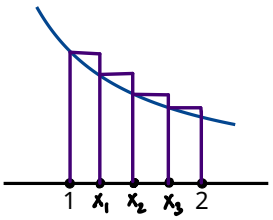
underestimation!

(b) Repeat (a) with left endpoints.

$L_4 = f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\Delta x$

$= \frac{1}{1}\cdot 0.25 + \frac{1}{1.25}\cdot 0.25 + \frac{1}{1.5}\cdot 0.25 + \frac{1}{1.75}\cdot 0.25$

≈ 0.7595



overestimation!

(c) **(do this with me)** Express the area of the region under the graph as a limit.

$a = 1, \quad b = 2, \quad \Delta x = \frac{b-a}{n} = \frac{1}{n}, \quad x_i = a + i\cdot\Delta x = 1 + \frac{i}{n}$

$R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n} \frac{1}{x_i}\cdot\Delta x = \sum_{i=1}^{n} \frac{1}{1+(i/n)}\cdot\frac{1}{n}$

$\text{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\cdot\frac{1}{1+(i/n)}$

Problem 2. The speed of a runner during the first 2 seconds of a race is increasing and measured at half second intervals, given in the table below. Find upper and lower estimates for the distance the runner traveled in these 3 seconds.

t (in seconds)	0	0.5	1	1.5	2	2.5	3
v (in ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

(**visualised here**)

distance = 0.5 × velocity in 0 − 0.5 s + 0.5 × velocity in 0.5 − 1 s
+ 0.5 × velocity in 1 − 1.5 s + 0.5 × velocity in 1.5 − 2 s
+ 0.5 × velocity in 2 − 2.5 s + 0.5 × velocity in 2.5 − 3 s

If the velocity in a 0.5 s period is the velocity at the beginning of this period

distance = 0.5 × 0 + 0.5 × 6.2
+ 0.5 × 10.8 + 0.5 × 14.9
+ 0.5 × 18.1 + 0.5 × 19.4

= 34.7 ft

If the velocity in a 0.5 s period is the velocity at the end of this period

distance = 0.5 × 6.2 + 0.5 × 10.8
+ 0.5 × 14.9 + 0.5 × 18.1
+ 0.5 × 19.4 + 0.5 × 20.2

= 44.8 ft

The Definite Integral. Let $f(x)$ be a function. The definite integral of f from a to b is provided that this limit exists.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

If it exists, we say f is *integrable* on $[a, b]$.

The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. So, That is, we could use any letter in place

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

of x without changing the value of the integral

Problem 3. Visit this Desmos link: https://www.desmos.com/calculator/rurbkpifdu. Approximate

$$\int_0^2 \frac{x}{x+1}\,dx$$

using 10, 50 and 100 subintervals and both right endpoints and left endpoints. Change $f(x)$ on Desmos. What does this tell you about the true value of the integral?

n	R_n	L_n	$R_n - L_n$
10	0.965	0.832	0.133
50	0.915	0.888	0.027
100	0.908	0.895	0.013

Value of integral is between 0.908 and 0.895

Example A. The following limit is determining an area under the curve of some function $f(x)$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cdot \frac{1}{1+(2i/n)} \longrightarrow \sum_{i=1}^{n} f(x_i)\cdot \Delta x$$

$a+i\cdot\Delta x$

Determine what it is as a definite integral. That is, what is a, b and $f(x)$?

So, $\Delta x = \underbrace{\frac{b-a}{n} = \frac{2}{n}}_{\circledast}$ $\quad f(x_i) = \frac{1}{1+(2i/n)}$

$\rightsquigarrow f(a+2i/n) = \frac{1}{1+(2i/n)}$

$\int_1^3 \frac{1}{x}\,dx$ OR $\int_0^2 \frac{1}{1+x}\,dx$

$\rightsquigarrow$ Guesstimate : $f(x)=\frac{1}{x}$, $a=1$ OR $f(x)=\frac{1}{1+x}$, $a=0$

using $\circledast$: $b=3$ $\qquad$ $b=2$

Problem 4. The limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1+x_i}\cdot\Delta x$ represents an integral $\int_0^1 f(x)\,dx$. What is $f(x)$?

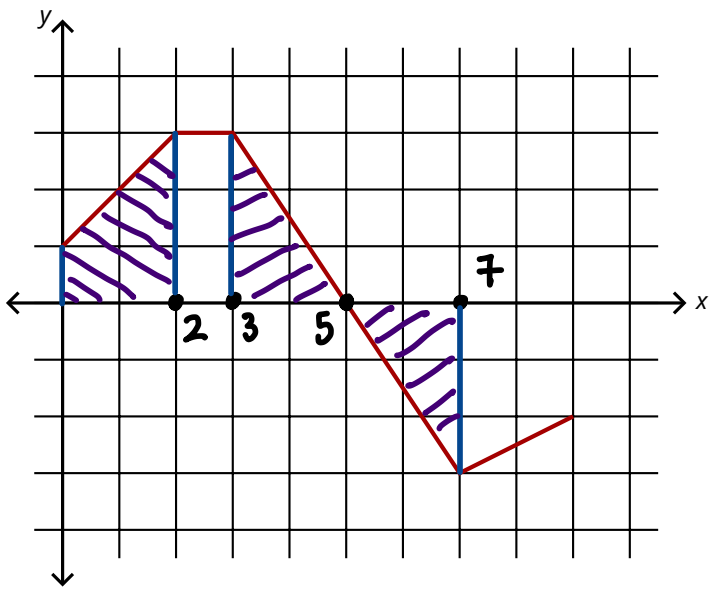
$\sum_{i=1}^{n} f(x_i)\Delta x$

So $f(x_i) = \frac{e^{x_i}}{1+x_i}$ $\rightsquigarrow$ $f(x) = \frac{e^x}{1+x}$

Problem 5. Shown below is the graph of f. Estimate each of the integrals by interpreting it as an area; remember to use function output as height.

$$\int_0^2 f(x)\,dx \qquad \int_5^7 f(x)\,dx \qquad \int_3^7 f(x)\,dx \qquad \int_2^0 f(x)\,dx$$

(**think about what our width should be and what our height should be**)



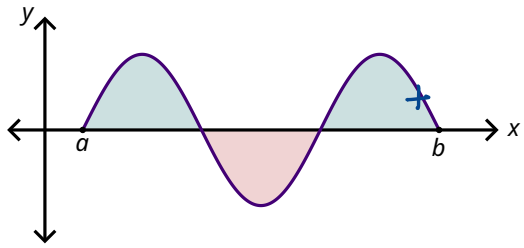
$$\int_0^2 f(x)\,dx = (2-0)\cdot 1 + \frac{1}{2}(2-0)\cdot 2 = 4$$

$$\int_5^7 f(x)\,dx = \frac{1}{2}\cdot(7-5)\cdot(-3) = -3$$

$$\int_3^7 f(x)\,dx = \text{area I} + \text{area II} = \frac{1}{2}(5-3)\cdot 3 + (-3) = 0$$

$$\int_2^0 f(x)\,dx = (0-2)\cdot 1 + \frac{1}{2}(0-2)\cdot 2 = -4$$

Computing Net Area. The computation $\int_a^b f(x)\,dx$ gives you



signed area

and therefore the resulting numerical value is net area

Computing Total Area

The **total area** under the graph of $y = f(x)$, between $x = a$ & $x = b$, is obtained by computing

$$\int_a^b |f(x)|\,dx$$

Properties of the Definite Integral. Let $f(x)$ be an integrable function between $x = a$ and $x = b$. Then,

Property 1. $\int_b^a f(x)\,dx = -\int_b^a f(x)\,dx$ since $\frac{a-b}{n} = -\frac{b-a}{n}$

Property 2. So, when $b = a$, we have $\int_a^a f(x)\,dx = 0$ since $\frac{a-a}{n} = 0$

If $g(x)$ is another integrable function between $x = a$ and $x = b$ and k is any constant, then

Property 3. $\int_a^b k\,dx = k(b-a)$

Property 4. $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

Property 5. $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Property 6. $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Let c be another number (traditionally $a \leqslant c \leqslant b$)

Property 7. $\int_a^c f(x)dx + \int_c^b f(x) = \int_a^b f(x)\,dx$

Property 8. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$

Property 9. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$

Name: Deew ang

Day 17. Antiderivatives

(based on Sections 4.9)

Learning Objectives

- Find antiderivative functions F of functions f like polynomial, power, exponential and trigonometric functions.
- Recognise antiderivatives as a family of functions; unique up to an arbitrary constant C.

Properties of the Definite Integral. Let $f(x)$ be an integrable function between $x = a$ and $x = b$. Then,

Property 1. $\int_b^a f(x)\,dx =$ $-\int_b^a f(x)\,dx$ since $\frac{a-b}{n} = -\frac{b-a}{n}$

Property 2. So, when $b = a$, we have $\int_a^a f(x)\,dx =$ 0 since $\frac{a-a}{n} = 0$

If $g(x)$ is another integrable function between $x = a$ and $x = b$ and k is any constant, then

Property 3. $\int_a^b k\,dx =$ $k\,(b-a)$

Property 4. $\int_a^b k \cdot f(x)\,dx =$ $k \cdot \int_a^b f(x)\,dx$

Property 5. $\int_a^b f(x) + g(x)\,dx =$ $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Property 6. $\int_a^b f(x) - g(x)\,dx =$ $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

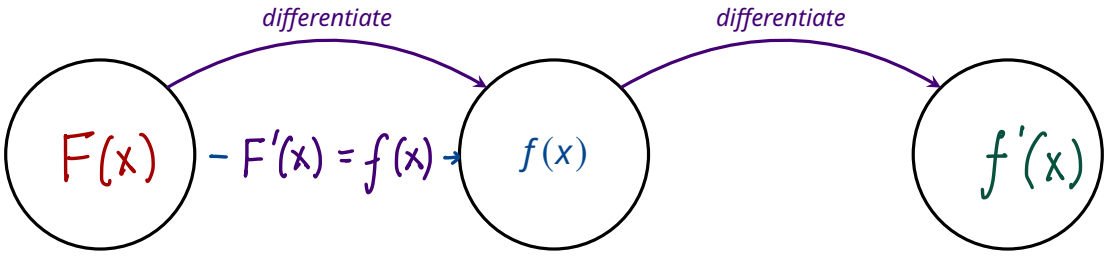
Let c be another number (traditionally $a \leqslant c \leqslant b$)

Property 7. $\int_a^c f(x)dx + \int_c^b f(x) =$ $\int_a^b f(x)\,dx$

Property 8. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx$ $\geqslant 0$

Property 9. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$

Antiderivatives



That is, given a function $f(x)$ we are attempting to fill in the following blank

$$\frac{d}{dx}\Big(\quad ? \quad\Big) = f(x)$$

The Antiderivative. We say $F(x)$ is an **antiderivative** of a function $f(x)$, on an interval, if

$$F'(x) = f(x)$$

for all x in that interval.

Example A. Find antiderivatives of the following functions

(a) $f(x) = 3x^2$

x^3, $x^3 + 1$

(b) $g(t) = -\sin(t)$

$\cos(t)$

(c) $h(u) = \ln(2) \cdot 2^u$

2^u, $2^u + 3$

(d) $k(r) = \dfrac{1}{2\sqrt{r}}$

$\sqrt{r}$

(e) $a(x) = \dfrac{1}{x}$

$\ln(x)$

(f) $f(x) = x^2$

$x^3/3$, $x^3/3 + 7$

(g) $g(t) = \sin(t)$

$-\cos(t)$

(h) $h(u) = 2^u$

$2^u / \ln(2)$

(i) $k(r) = \dfrac{1}{\sqrt{r}}$

$2\sqrt{r}$, $2\sqrt{r} + 11$

(j) $a(x) = \dfrac{1}{\ln(5) \cdot x}$

$\dfrac{\ln(x)}{\ln(5)} = \log_5(x)$

Problem 1. Find antiderivatives of the following functions

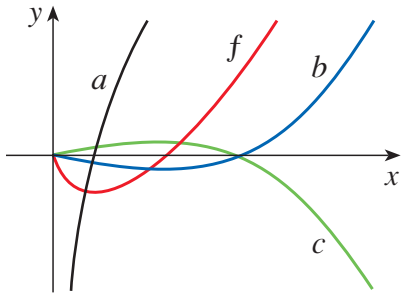
(a) $f(x) = 4x^3 + 2x + 1$ $x^4 + x^2 + x$

(b) $g(t) = e^t + \cos(t)$ $e^t + \sin(t)$

(c) $k(x) = \frac{1}{\sqrt{x}} + \frac{1}{x}$ $2\sqrt{x} + \ln(x)$

(d) $f(x) = x^3 + x + 3$ $\frac{x^4}{4} + \frac{x^2}{2} + 3x$

Problem 2. Which one of the following graphs is an antiderivative of f?



(b)

The Most General Antiderivative. If F is an antiderivative of f on an interval, its **most general antiderivative** is

$$F(x) + C$$

where C is any constant.

Slogan: *antiderivatives are unique up to a constant*

Example B. Let $f(x) = \sin(x)$.

(a) List three antiderivatives of $f(x)$ and sketch them. $-\cos(x)$, $-\cos(x)+1$, $-\cos(x)-2$

(b) List all its antiderivatives. $-\cos(x) + C$

(c) Find the antiderivative $F(x)$ such that $F(0) = 2$.

$F(x) = -\cos(x) + C$

$2 = F(0) = -\cos 0 + C = -1 + C$

$C = 3$ $F(x) = -\cos(x) + 3$

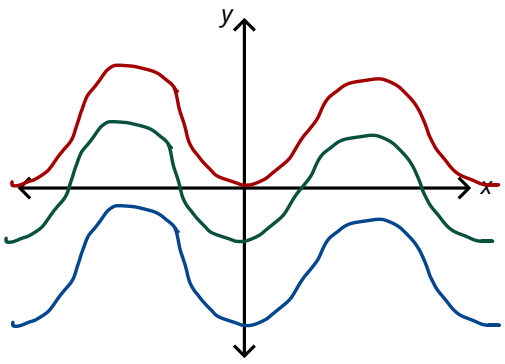


Table of General Antiderivatives

Assume an antiderivative of $f(x)$ is $F(x)$ and of $g(x)$ is $G(x)$, while a and C are constants

Function	**General Antiderivative**	*Remember*	*will be given*
$a \cdot f(x)$	$a \cdot F(x) + C$	!	
$f(x) \pm g(x)$	$F(x) \pm G(x) + C$	!	
x^n (when $n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$	!	
$x^{-1} = \frac{1}{x}$	$\ln(x) + C$	!	
e^x	$e^x + C$	!	
a^x	$\frac{a^x}{\ln(a)} + C$	!	
$\cos(x)$	$\sin(x) + C$	!	
$\sin(x)$	$-\cos(x) + C$	!	
$\sec^2(x)$	$\tan(x) + C$		✓
$\csc^2(x)$	$-\cot(x) + C$		✓
$\sec(x)\tan(x)$	$\sec(x) + C$		✓
$\csc(x)\cot(x)$	$-\csc(x) + C$		✓

Example C. What is an antiderivative of a constant?

What when differentiated gives $f(x) = C$ ⇝ Cx

What when differentiated gives $g(t) = C$ ⇝ Ct

Problem 3. Find f that satisfies the given conditions.

Refer to the table in the previous page to figure out antiderivatives of each term; do not forget +C

(a) $f'(t) = \sqrt{t} - 2$ with $f(9) = 4$.

$$f'(t) = t^{1/2} - 2 \cdot t^0$$

$$f(t) = \frac{t^{1/2+1}}{\frac{1}{2}+1} - 2 \cdot \frac{t^{0+1}}{0+1} + C = \frac{2}{3} \cdot t^{3/2} - 2t + C$$

$$= \frac{2}{3} t^{3/2} - 2t + 4$$

$$4 = f(9) = 18 - 18 + C \rightsquigarrow C = 4$$

(b) $f''(\theta) = \sin\theta + \cos\theta$ with $f(0) = 3$ and $f'(0) = 4$.

use different letters for +C when finding antiderivative twice

$$f'(\theta) = -\cos\theta + \sin\theta + C$$

$$4 = f'(0) = -1 + C \rightsquigarrow C = 5$$

$$f(\theta) = -\sin\theta - \cos\theta + C\theta + D$$

$$3 = f(0) = -1 + D \rightsquigarrow D = 4$$

$$f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$$

(c) $g''(t) = \frac{1}{t^2}$, where $t > 0$, with $g'(1) = 1$ and $g(1) = 0$

$g''(t) = t^{-2}$

$g'(t) = \frac{t^{-2+1}}{-2+1} + C = -\frac{1}{t} + C$

$1 = g'(1) = -1 + C \rightsquigarrow C = 2$

$g(t) = -\ln(t) + Ct + D$

$0 = g(1) = C + D \rightsquigarrow D = -2$

$g(t) = -\ln(t) + 2t - 2$ ⭐

Recap Problems

The aim is to review Day 16 material; so have the filled-in Day 16 notes in front of you as a reference!

Evaluate the given integrals by interpreting them in terms of areas by following the given steps.

1. Sketch the graph of the given function you are integrating and shade the area you have been asked to compute. You may use Desmos for this, but reproduce a sketch below.
2. Identify the shaded area with area(s) of familiar shapes.
3. Compute the area using the formulae for areas of the identified shapes.

Recap Problem 1. $\int_3^5 2x - 3\,dx$

Recap Problem 2. $\int_0^2 \sqrt{4-(x-2)^2}\,dx$

- Area of triangle with base b and height h is $\frac{1}{2}bh$.
- Area of a circle with radius r is πr^2.
- Area of a rectangle with length ℓ and width w is ℓw.

Name: Deewang

Day 18. Fundamental Theorem(s) of Calculus

(based on Sections 5.3)

Learning Objectives

- Evaluate definite integrals using the Fundamental Theorem of Calculus.
- Differentiate functions expressed as integrals using the Fundamental Theorem of Calculus.

Let us return to our discussion on area and the definite integral. The following theorem shows us how antiderivatives help us compute definite integrals!

Fundamental Theorem of Calculus 2 (FTC 2). Let $f(x)$ be a continuous function on the interval $[a, b]$. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$$

where F is an antiderivative of f $\rightsquigarrow$ $F' = f$

Example A. Let us compute

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

an antiderivative of x^2 is $\frac{x^3}{3}$

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Problem 1. Compute the following. Refer to the table of antiderivatives from the previous class, e.g., on Moodle.

(a) $\int_0^1 x^2\,dx$, using a different antiderivative

$$= \left[\frac{x^3}{3} + 2\right]_0^1 = \left(\frac{1^3}{3} + 2\right) - \left(\frac{0^3}{3} + 2\right) = \frac{1}{3}$$

(b) $\int_{-1}^1 2x^3 + 5\,dx = \left[2 \cdot \frac{x^4}{4} + 5x\right]_{-1}^1$

$$= \left[\frac{x^4}{2} + 5x\right]_{-1}^1 = \left(\frac{1^4}{2} + 5 \cdot 1\right) - \left(\frac{(-1)^4}{2} - 5(-1)\right) = 10$$

(c) $\int_{-2}^{-1} \frac{6}{x^4}\,dx = \int_{-2}^{-1} 6 \cdot x^{-4}\,dx$

$= \left[6 \cdot \frac{x^{-3}}{-3}\right]_{-2}^{-1} = \left[\frac{-2}{x^3}\right]_{-2}^{-1} = \frac{-2}{(-1)^3} - \frac{-2}{(-2)^3} = \frac{7}{4}$

(d) $\int_{\pi/2}^{0} \cos(\theta)\,d\theta$

$= \left[\sin\theta\right]_{\pi/2}^{0} = \sin 0 - \sin\frac{\pi}{2} = -1$

(e) $\int_{1}^{e^2} \frac{1}{u}\,du$

$= \left[\ln u\right]_{1}^{e^2} = \ln e^2 - \ln 1 = 2$

(f) $\int_{0}^{-1} (1+r)^{10}\,dr$

$= \left[\frac{(1+r)^{11}}{11}\right]_{0}^{-1} = \frac{(1-1)^{11}}{11} - \frac{(1+0)^{11}}{11} = -\frac{1}{11}$

(g) $\int_{-1}^{3} \frac{1}{x^2}\,dx$

$= \left[-\frac{1}{x}\right]_{-1}^{3} = -\frac{1}{3} - \frac{-1}{-1} = -\frac{4}{3}$ WAIT !!!

see (h)

(h) Are you sure about your answer to (g)? Refer back to FTC 2 and explain why it doesn't apply

The function $\frac{1}{x^2}$ is not continuous on $[-1,3]$ since $\frac{1}{x^2}$ is not defined at 0

Our "answer" is wrong!

Net Change Theorem. The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example B. If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or **displacement**, of the object during the time period t_1 to t_2.

Problem 2. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does

$$100 + \int_0^{15} n'(t)\,dt$$

represent?

By Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0)$

We are given $n(0) = 100$. So, $100 + \int_0^{15} n'(t)\,dt = n(15)$

What is an antiderivative of e^{-x^2}?

No clean answer! We have to "construct" it!

Fundamental Theorem of Calculus 1 (FTC 1). Let $f(x)$ be a continuous function on the interval $[a,b]$. Define

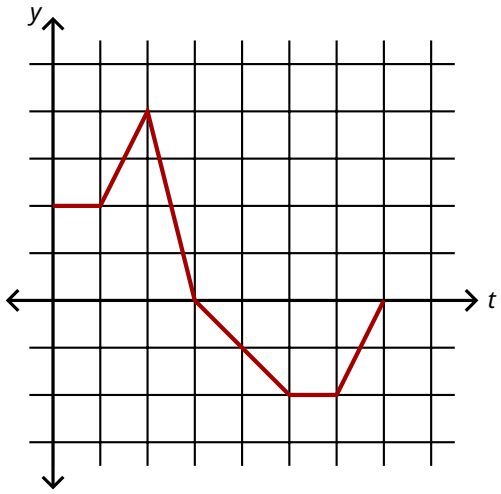
$$g(x) = \int_a^x f(t)\,dt \qquad \text{then } g'(x) = f(x)$$

$g(x)$ is an antiderivative of $f(x)$

a can be anything in the domain of f. It plays no role.

This theorem is also called the **Construction Theorem of Calculus**.

Problem 3. Let $g(x) = \int_0^x f(t)\,dt$, where the graph of f is given below.



(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$ and $g(6)$ by interpreting them as area.

$$g(0) = \int_0^0 f(t)\,dt = 0$$

$$g(1) = \int_0^2 f(t)\,dt = 2$$

$$g(2) = \int_0^2 f(t)\,dt = 5$$

$$g(3) = \int_0^3 f(t)\,dt = 7$$

$$g(6) = \int_0^6 f(t)\,dt$$

$$= 7 - 4 = 3$$

upper area ← → lower area

(b) On what interval is $g(x)$ increasing? On what interval is it decreasing?

$g(x)$ is increasing on $[0,3]$ since it is accumulating positive area

$g(x)$ is decreasing on $[3,7]$ since it is accumulating negative area

(c) For which x does g have a maximum value?

$g(x)$ reaches max at $x = 3$

Example C. Compute the derivatives of the following functions, using FTC 1.

(a) $g(x) = \int_0^x \sqrt{t^2 + t^3}\, dt$

$g'(x) = \sqrt{x^2 + x^3}$

(b) $f(x) = \int_5^x \sqrt{t^2 + t^3}\, dt$

$f'(x) = \sqrt{x^2 + x^3}$. the beginning end-point (a constant) does not matter

(c) $h(x) = \int_x^2 \sqrt{t^2 + t^3}\, dt$

$= -\int_2^x \sqrt{t^2 + t^3}\, dt$ $\rightsquigarrow$ $h'(x) = -\sqrt{x^2 + x^3}$

(d) $k(x) = \int_0^{x^2} \sqrt{t^2 + t^3}\, dt$

Define $a(x) = \int_0^x \sqrt{t^2 + t^3}\, dt$, then $k(x) = a(x^2)$

$\rightsquigarrow$ $a'(x) = \sqrt{x^2 + x^3}$ and by chain rule

$k'(x) = a'(x^2) \cdot 2x = \sqrt{(x^2)^2 + (x^2)^3} \cdot 2x$

$= 2x\sqrt{x^4 + x^6}$

Problem 4. Compute the following using FTC 1.

(a) $\frac{d}{dx}\left[\int_5^x \frac{t^5+t^4+17}{(t^3+1)^{3/2}}\,dt\right]$

$$= \frac{x^5+x^4+17}{(x^3+1)^{3/2}}$$

(b) $\frac{d}{dx}\left[\int_x^0 \frac{\sin(t)}{t}\,dt\right]$

$$= \frac{d}{dx}\left[-\int_0^x \frac{\sin(t)}{t}\,dt\right] = -\frac{d}{dx}\left[\int_0^x \frac{\sin(t)}{t}\,dt\right] = -\frac{\sin(x)}{x}$$

(c) $\frac{d}{dx}\left[\underbrace{\int_1^{\sin(x)} t^2\,dt}_{g(\sin x)}\right]$ where $g(x) = \int_1^x t^2\,dt \rightsquigarrow g'(x) = x^2$

$$= \frac{d}{dx}\left(g(\sin x)\right) = g'(\sin(x))\cdot \underbrace{\cos x}_{(\sin x)'} = \sin^2(x)\cdot\cos(x)$$

(d) $\frac{d}{dx}\left[\underbrace{\int_0^{e^x-1} \ln(t+1)\,dt}_{g(e^x-1)}\right]$ where $g(x) = \int_0^x \ln(t+1)\,dt \rightsquigarrow g'(x) = \ln(x+1)$

$$= \frac{d}{dx}\left(g(e^x-1)\right) = g'(e^x-1)\cdot \underbrace{e^x}_{(e^x-1)'} = \ln(e^x)\cdot e^x = x\cdot e^x$$

Recap Problems

The aim is to review Day 17 material; so have the filled-in Day 17 notes in front of you as a reference!

Recap Problem 1. Find $f(x)$ such that $f'(x) = 2e^x + \sin(x)$ and $f(0) = -1$

Recap Problem 2. $g''(t) = \dfrac{-2}{t^2} + 3$, where $t > 0$, with $g'(1) = 2$ and $g(1) = 0$

Table of General Antiderivatives

Assume an antiderivative of $f(x)$ is $F(x)$ and of $g(x)$ is $G(x)$, while a and C are constants

Function	**General Antiderivative**	*Remember*	*will be given*
$a \cdot f(x)$	$a \cdot F(x) + C$	!	
$f(x) \pm g(x)$	$F(x) \pm G(x) + C$	!	
x^n (when $n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$	!	
$x^{-1} = \frac{1}{x}$	$\ln(x) + C$	!	
e^x	$e^x + C$	!	
a^x	$\frac{a^x}{\ln(a)} + C$	!	
$\cos(x)$	$\sin(x) + C$	!	
$\sin(x)$	$-\cos(x) + C$	!	
$\sec^2(x)$	$\tan(x) + C$		✓
$\csc^2(x)$	$-\cot(x) + C$		✓
$\sec(x)\tan(x)$	$\sec(x) + C$		✓
$\csc(x)\cot(x)$	$-\csc(x) + C$		✓

Name: Deewang

Day 19. Indefinite Integrals & Method of Substitution

(based on Sections 5.5)

Learning Objectives

- Work with indefinite integrals.
- Evaluate definite and indefinite integrals using substitution.

We first introduce a convenient notation for the most general antiderivative of a given function $f(x)$.

The Indefinite Integral. Given $f(x)$, its most general antiderivative is written as

$$\int f(x)\,dx$$

and called the **indefinite integral** of $f(x)$.

$$F(x) = \int f(x)\,dx \quad \text{then} \quad F'(x) = f(x)$$

Put another way, if $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C \qquad C \text{ a constant}$$

Warning. For a function $f(x)$, we now have two different integrals:

- The **definite integral** $\int_a^b f(x)\,dx$ is a **number**.
- The **indefinite integral** $\int f(x)\,dx$ is a **function**. In fact, it is a *family* of functions.

Example A. What is $\int 6x^2 - 5\dfrac{\sin(x)}{\cos^2(x)}\,dx$?

$$= \int 6x^2 - 5 \cdot \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}\,dx$$

$$= \int 6x^2 - 5\sec(x)\cdot\tan(x)\,dx$$

$$= 6\cdot\frac{x^3}{3} - 5\sec(x) + C$$

Problem 1. Compute $\int \dfrac{1}{\cos^2(\theta)} + 4\theta^3 + e^\theta\,d\theta$

$$= \int \sec^2(\theta) + 4\theta^3 + e^\theta\,d\theta$$

$$= \tan(\theta) + \theta^4 + e^\theta + C$$

Table of Indefinite Integrals

a and C are constants

Indefinite Integrals	*Remember*	*will be given*
$\int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx$	!	
$\int f(x) \pm g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$	!	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (\text{when } n \neq -1)$	!	
$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(x) + C$	!	
$\int e^x\,dx = e^x + C$	!	
$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$	!	
$\int \cos(x)\,dx = \sin(x) + C$	!	
$\int \sin(x)\,dx = -\cos(x) + C$	!	
$\int \sec^2(x)\,dx = \tan(x) + C$		✓
$\int \csc^2(x)\,dx = -\cot(x) + C$		✓
$\int \sec(x)\tan(x)\,dx = \sec(x) + C$		✓
$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$		✓

Substitution Rule for Indefinite Integrals. If $u = g(x)$ is a differentiable function whose range is an interval and f is continuous on that interval, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

This is using the chain rule as a technique. Since if F is an antiderivative of f, then

$$\int F'(g(x))\, g'(x)\, dx = \int f(g(x))\, g'(x)\, dx$$

$$= \int f(u)\, du$$

$$= F(u) + C = F(g(x)) + C$$

$F' = f$

Example B. Compute $\int x^2 e^{x^3}\, dx$

Step 1. Identify your u to substitute, u is your "inside function".

$u = x^3 \leftarrow g(x)$

Step 2. Let $u = g(x)$, compute du using the formula $du = g'(x)\, dx$

$du = 3x^2\, dx$

Step 3. Solve for dx.

$dx = \frac{1}{3x^2}\, du$

Step 4. Go back to your integral re-write your function in terms of u and dx in terms of du.

Everything should be now in terms of *u* or *du*, there should be nothing involving *x*.

$$\int x^2 e^{x^3}\, dx = \int x^2 e^u \cdot \frac{1}{3x^2}\, du = \int \frac{1}{3} e^u\, du$$

$$= \frac{1}{3} \cdot \int e^u\, du$$

Step 5. Use standard antiderivatives to compute your indefinite integral as a function in u.

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C$$

Step 6. Substitute back $u = g(x)$ to obtain your final answer, do not forget the $+C$!

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$$

Substituting back $u = x^3$

Example C. Compute $\int x\sqrt{2x+7}\,dx$

Step 1. Identify your u to substitute, u is your "inside function".

$u = 2x + 7 \leftarrow g(x)$

Step 2. Let $u = g(x)$, compute du using the formula $du = g'(x)\,dx$

$du = 2\,dx$

Step 3. Solve for dx.

$dx = \frac{1}{2}\,du$

Step 4. Go back to your integral re-write your function in terms of u and dx in terms of du.

Everything should be now in terms of *u* or *du*, there should be nothing involving *x*.

$$\int x\sqrt{2x+7}\,dx = \int x\cdot\sqrt{u}\cdot\frac{1}{2}\,du = \int \frac{u-7}{2}\cdot\sqrt{u}\cdot\frac{1}{2}\,du$$

???

Go back to $u = 2x+7$

Solve for x in terms of u

$$= \frac{1}{4}\int (u-7)\sqrt{u}\,du$$

Step 5. Use standard antiderivatives to compute your indefinite integral as a function in u.

$$\frac{1}{4}\int (u-7)\sqrt{u}\,du = \frac{1}{4}\int u^{3/2} - 7\cdot u^{1/2}\,du$$

$$= \frac{1}{4}\left(\frac{2}{5}\cdot u^{5/2} - 7\cdot\frac{2}{3}u^{3/2}\right) + C$$

Step 6. Substitute back $u = g(x)$ to obtain your final answer, do not forget the $+C$!

$$\int x\sqrt{2x+7}\,dx = \frac{1}{10}(2x+7)^{5/2} - \frac{7}{6}(2x+7)^{3/2} + C$$

Problem 2. Use substitution with the u provided to evaluate the given indefinite integral.

(a) $\int \cos(2x)\,dx$ with $u = 2x$

$du = 2\,dx \rightsquigarrow dx = \frac{1}{2}du$

$$\int \cos(2x)\,dx = \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C$$

$$= \frac{1}{2}\sin(2x) + C$$

(b) $\int \sin^2(\theta)\cos(\theta)\,d\theta$ with $u = \sin(\theta)$

$du = \cos\theta\,d\theta$

$$\int \sin^2(\theta)\cos(\theta)\,d\theta = \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\sin^3(\theta)}{3} + C$$

Problem 3. Use substitution to evaluate the given indefinite integral. You choose the u.

(a) $\int t^3\sqrt{1+t^4}\,dt$ $\quad u = 1+t^4$

$du = 4t^3\,dt \rightsquigarrow t^3\,dt = \frac{1}{4}du$

$$\int t^3\sqrt{1+t^4}\,dt = \frac{1}{4}\int \sqrt{u}\,du$$

$$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{6}(1+t^4)^{3/2} + C$$

(b) $\int \frac{dx}{4x+7}$ $\quad u = 4x+7$

$du = 4\,dx \rightsquigarrow dx = \frac{1}{4}du$

$$\int \frac{dx}{4x+7} = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln(u) + C$$

$$= \frac{1}{4}\ln(4x+7) + C$$

Substitution Rule for Definite Integrals. If $g(x)$ is differentiable and g' is continuous on $[a, b]$, and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example D. Compute $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

Step 1. Identify your u to substitute, u is your "inside function".

$u = 1/x$ ← $g(x)$

Step 2. Compute your **new bounds**.

$a = 1 \rightsquigarrow g(a) = 1$, $b = 2 \rightsquigarrow g(b) = 1/2$

Step 3. Let $u = g(x)$, compute du using the formula $du = g'(x)\, dx$ and solve for dx.

$du = -\frac{1}{x^2} dx \rightsquigarrow \frac{1}{x^2} dx = -du$

Step 4. Go back to your integral re-write your function in terms of u and dx in terms of du, and **use the new bounds**. Solve the resulting integral

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^{1/2} -e^u\, du = \left[-e^u\right]_1^{1/2} = -e^{1/2} + e$$

do NOT plug back u, use new bounds!

Warning. Observe that there is a major difference between using u-substitution for indefinite and definite integrals.

- In the steps for indefinite integration using u-substitution, you *substitute back your u at the very end.*
- In the steps for definite integration using u-substitution, you *DO NOT substitute back your u at the very end.* Instead, *you are computing new bounds at the beginning and then integrating a definite integral in terms of u.*

  Therefore, **DO NOT substitute back *u* while computing a definite integral!**

Problem 4. Compute.

(a) $\int_0^1 t(t^2+1)^3\,dt$

$u = t^2+1 \leftarrow g(t)$

New bounds: $0 \rightsquigarrow g(0) = 1$

$1 \rightsquigarrow g(1) = 2$

$du = 2t\,dt \rightsquigarrow t\,dt = \frac{1}{2}du$

$$\int_0^1 t(t^2+1)^3\,dt = \frac{1}{2}\int_1^2 u^3\,du$$

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{15}{8}$$

(b) $\int_{\pi^2/9}^{\pi^2/4} \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx.$ $\sin\left(\frac{\pi}{2}\right) = 1$ & $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$u = \sqrt{x} \leftarrow g(x)$

New bounds:

$\pi^2/9 \rightsquigarrow g(\pi^2/9) = \pi/3$

$\pi^2/4 \rightsquigarrow g(\pi^2/4) = \pi/2$

$du = \frac{1}{2\sqrt{x}}\,dx \rightsquigarrow \frac{1}{\sqrt{x}}\,dx = 2\,du$

$$\int_{\pi^2/9}^{\pi^2/4} \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = 2\int_{\pi/3}^{\pi/2} \cos(u)\,du$$

$$= 2\left[\sin(u)\right]_{\pi/3}^{\pi/2} = 2-\sqrt{3}$$

(c) $\int_1^e \frac{(\ln(t))^2}{t}\,dt$

$u = \ln(t) \leftarrow g(t)$

New bounds: $1 \rightsquigarrow g(1) = 0$

$e \rightsquigarrow g(e) = 1$

$du = \frac{1}{t}\,dt$

$$\int_1^e \frac{(\ln(t))^2}{t}\,dt = \int_0^1 u^2\,du$$

$$= \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

(d) $\int_0^4 \frac{2x^3}{\sqrt{x^2+9}}\,dx$

$u = x^2+9 \leftarrow g(x)$

$x^2 = u-9$

New bounds

$0 \rightsquigarrow g(0) = 9$

$4 \rightsquigarrow g(4) = 25$

$du = 2x\,dx$

$$\int_0^4 \frac{2x^3}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{u-9}{\sqrt{u}}\,du$$

$$= \int_9^{25} u^{1/2} - 9u^{-1/2}\,du$$

$$= \left[\frac{2}{3}u^{3/2} - \frac{9}{2}u^{1/2}\right]_9^{25} = \frac{88}{3}$$

Recap Problems

The aim is to review Day 18 material; so have the filled-in Day 18 notes in front of you as a reference!

Recap Problem 1. Express the following limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2x_i^2 + \sqrt{x_i}}{\sqrt{x_i}} \cdot \Delta x \qquad \text{on the interval } [1, 4]$$

and then compute the definite integral.

Recap Problem 2. Let $g(x) = \displaystyle\int_{x^2}^{0} -\ln(t)\, dt$. Compute $g'(x)$.

Name: Deewang

Day 21. Local/Absolute Maxima & Minima

(based on Sections 4.1)

Learning Objectives

- Identify local and absolute maximum and minimum values of a function.

Absolute Maxima and Minima. Let c be a number in the domain of f. Then,

- $f(c)$ is the **absolute maximum of f**, or the **absolute maximum of f is at c** if

$$f(c) \geq f(x)$$

for every x in the domain of f.

- $f(c)$ is the **absolute minimum of f**, or the **absolute minimum of f is at c** if

$$f(c) \leq f(x)$$

for every x in the domain of f.

Local Maxima and Minima. Let c be a number in the domain of f. Then,

- $f(c)$ is a **local maximum of f**, or a **local maximum of f is at c** if

$$f(c) \geq f(x)$$

when x *is near* c (both from left and right).

- $f(c)$ is a **local minimum of f**, or a **local minimum of f is at c** if

$$f(c) \leq f(x)$$

when x *is near* c (both from left and right).

Important. Local Maxima/Minima can only occur at points where the function is still defined to the left and right of those points. This does not apply to absolute maxima/minima.

Problem 1. Use the graph below to find the absolute and local maximum and minimum values of f.

Local Maxima

$f(4)$, $f(6)$

Local Minima

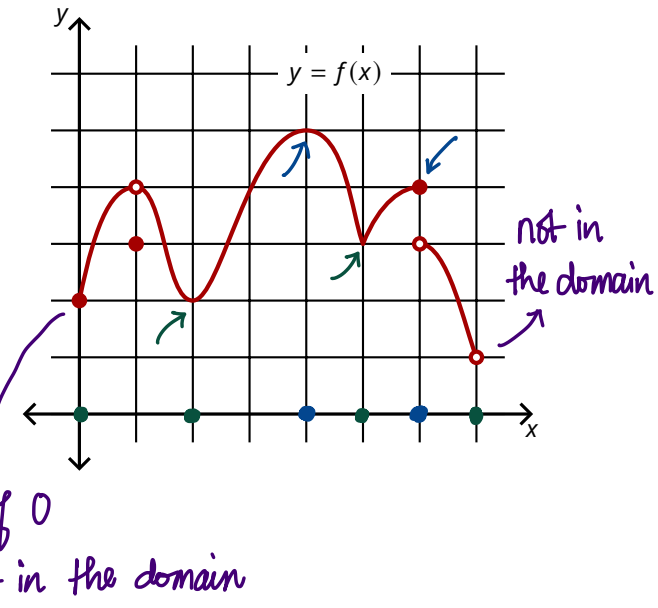
$f(2)$, $f(5)$

Absolute Maxima

$f(4)$

Absolute Minima

D.N.E.



Problem 2. Sketch the graph of a function $f(x)\ldots$

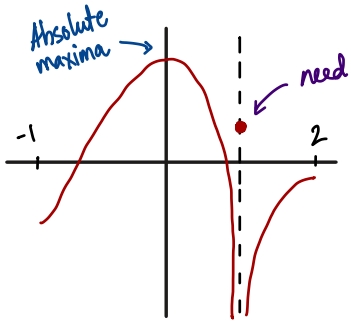
(a) that has a local maximum at 2 but is not continuous at 2.

Recall that not being continuous at 2 either means $\lim_{x\to 2} f(x)$ does not exist or $\lim_{x\to 2} f(x) \neq f(2)$.



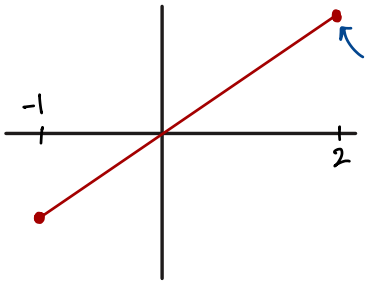
not cts at 2 since $\lim_{x\to 2} f(x)$ d.n.e. as $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

(b) on $[-1, 2]$ that has an absolute maximum but no absolute minimum.



need this since the function needs to be defined on every point in $[-1,2]$

(c) on $[-1, 2]$ that has an absolute maximum but no local maximum.



No local max/min since they cannot occur at end points

"near c" means "near, on both sides of, c"

Critical Numbers. A **critical number** of a function f is a number c in the domain of f such that either

$f'(c) = 0$ or $f'(c)$ does not exist

Problem 3. Find critical numbers of the following functions:

(a) $f(x) = x^3$ $f'(x) = 3x^2$

$0 = f'(c) = 3c^2 \rightsquigarrow c = 0$ is a critical no.

(b) $f(x) = x^{1/3}$ $f'(x) = \frac{1}{3 \cdot x^{2/3}}$

$f'(x)$ does not exist for $x = 0 \rightsquigarrow c = 0$ is a critical no.

(c) $f(x) = x^4 + \ln(x)$

$f'(x) = 4x^3 + \frac{1}{x}$

$0 = f'(c) = 4c^3 + \frac{1}{c} \rightsquigarrow c^4 = -\frac{1}{4}$ no soln $\rightsquigarrow$ No critical no.

(d) $f(x) = |x|$

$f'(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

$f'(x)$ does not exist for $x=0$

$\rightsquigarrow c=0$ is a critical no.

Local Maxima/Minima are Critical Numbers

If f has a local maxima or minima at c

then c is a critical number

Problem 4. Look back at **Problem 3.** For the critical numbers you found for those functions, which of those correspond to a local maxima or minima?

x^3 and $x^{1/3}$ have $c=0$ as a critical no. but it does not correspond to a local max/min

$|x|$ has $c=0$ as a critical no. and there is a local min at $c=0$.

Problem 5. Find the critical numbers of the function and speculate on whether they correspond to an absolute maximum, absolute minimum, or neither. Then graph the function (e.g., on Desmos) and check your speculation.

(a) $f(x) = x^3 - 12x + 4$

$f'(x) = 3x^2 - 12 \leftarrow 3x^2 - 12 = 0 \rightsquigarrow x = -2, 2$ are critical no.

$x=-2$ ← local max @

$x=2$ ← local min @

(b) $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b \leftarrow 2ax + b = 0 \rightsquigarrow x = -b/2a$ is a critical no.

absolute max/min @

(c) $f(x) = |x - 6|$

$$f'(x) = \begin{cases} 2 & x > 6 \\ -2 & x < 6 \end{cases}$$

does not exist at $x = 6$ ← critical no.

absolute minimum @

(d) $f(x) = \dfrac{x^2 + 2}{2x - 1}$

$$f'(x) = \frac{2x(2x-1) - (x^2+2)(2)}{(2x-1)^2} = \frac{2x^2 - 2x - 4}{(2x-1)^2}$$

$$\rightsquigarrow \frac{2x^2-2x-4}{(2x-1)^2} = 0 \rightsquigarrow 2x^2 - 2x - 4 = 0$$

$$\rightsquigarrow 2(x+1)(x-2) = 0$$

$\rightsquigarrow$ $x = -1$ are critical no.

$x = 2$

local max @ $x = -1$

local min @ $x = 2$

Closed Interval Method.

To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$.

1. Compute critical numbers of f in $[a, b]$, and find the values of f at these critical numbers.
2. Find the values of f at the endpoints of the interval, i.e., $f(a)$ and $f(b)$.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example A. Find the absolute maximum and minimum values of $f(x) = x^2(x^2 - 4)$ on $[-1, 3]$.

$$f(x) = x^4 - 4x^2 \rightsquigarrow f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$$

$f'(x) = 0$ when $x = 0$, $x = \sqrt{2}$, $x = -\sqrt{2}$ } critical no.

≈ 1.414 ≈ -1.414

In $[-1, 3]$? ✓ ✓ ✗

$f(0) = 0$, $f(\sqrt{2}) = -4$, $f(-1) = -3$, $f(3) = 45$

abs. min in $[-1, 3]$ abs. max in $[-1, 3]$

Problem 6. Find the absolute maximum and minimum values of f on the given interval, following the steps above.

(a) $f(x) = x \cdot (x^2 - 3x - 24)$ on $[0, 5]$.

$= x^3 - 3x^2 - 24x$

$f'(x) = 3x^2 - 6x - 24$

$= 3(x+2)(x-4)$

Critical no.: $x = -2$, $x = 4$

In $[0,5]$? ✗ ✓

$f(0) = 0$, $f(4) = -80$, $f(5) = -70$

absolute max absolute min in $[0,5]$

(b) $f(x) = x + 1/x$ on $[0.2, 4]$.

$f'(x) = 1 - 1/x^2 = \frac{x^2-1}{x} = \frac{(x+1)(x-1)}{x}$

Critical no.: $x = -1$, $x = 1$, $x = 0$

In $[0.2, 4]$? ✗ ✓ ✗

$f(0.2) = 5.2$, $f(1) = 2$, $f(4) = 4.25$

absolute max absolute min in $[0.2, 4]$

Problem 7. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function where the time t is measured in hours and C is measured in μg/mL

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

What is the maximum concentration of the antibiotic during the first 12 hours? means $0 \le t \le 12$

$C'(t) = 8(-0.4\,e^{-0.4t} + 0.6\,e^{-0.6t}) = 0$

$\rightsquigarrow e^{0.2t} = \frac{0.6}{0.4} = 1.5 \rightsquigarrow t = \frac{\ln(1.5)}{0.2} \approx 2.03$

Critical no.: $\frac{\ln(1.5)}{0.2} \approx 2.03$ hours in $[0, 12]$

$C\left(\frac{\ln(1.5)}{0.2}\right) \approx 1.19$, $C(0) = 0$, $C(12) \approx 0.06$

abs. max abs min.

Recap Problems

The aim is to review Day 19 material; so have the filled-in Day 19 notes in front of you as a reference!

Recap Problem 1. Determine the indefinite integral $\int x(1-x)^{2/3}\,dx$

Remember to re-write everything in terms of *u* before integrating. It will be useful think what is *x* in terms of your *u*.

Recap Problem 2. Compute the definite integral $\int_0^{\pi/3} \tan(x)\,dx$. Start by expressing $\tan(x)$ in terms of $\sin(x)$ and $\cos(x)$.

$$\sin(0) = 0,\ \cos(0) = 1,\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

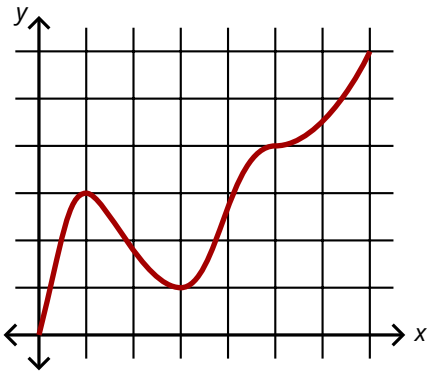
Name: Deewang

Day 24. What Derivatives tell us about the Shape of a Graph

(based on Sections 4.3)

- Identify intervals where a function is increasing or decreasing using the first derivative.
- Find local maximums and minimums using the First Derivative Test.
- Identify intervals of concavity and inflection points using the second derivative.
- Use the Second Derivative Test to classify local extrema.

Problem 1. The graph of $f(x)$ is given below. Using the graph, answer the following questions.



(a) Find the open intervals on which f is increasing.

$(0,1), (3,5), (5,7)$

(b) Find the open intervals on which f is decreasing.

$(1,3)$

Increasing/Decreasing Test. The first derivative of a function $f(x)$ tells us the **direction** of the function.

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Recall that **critical numbers** of a function $f(x)$ are those numbers c in the domain where

- $f'(c) = 0$ horizontal tangent line
- $f'(c)$ does not exist vertical tangent lines & corners

If f has a local max or min at c, then c must be a critical number; **but not every critical number gives rise to a max or a min**. We therefore need a test that will tell us whether or not f has a local max or min at a critical number.

First Derivative Test for Local Max/Min. Let c be a critical number of a continuous function $f(x)$.

- If f' **changes from positive on the left of c to negative on the right of c** then f has a local maximum **at c**.

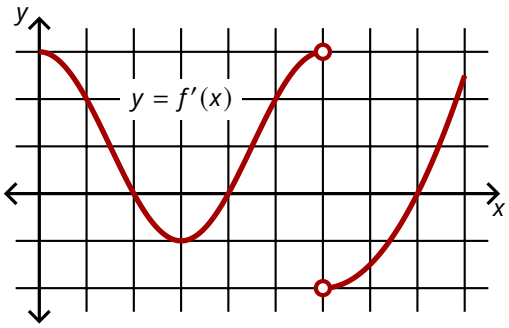


- If f' **changes from negative on the left of c to positive on the right of c** then f has a local minimum **at c**.



- If f' **does not change sign near c**, then f has neither a local max nor min **at c**.

Problem 2. Given below is the graph of the *derivative* $f'(x)$ of some function $f(x)$.



(a) What are the critical numbers of f?

where the derivative either **equals 0** *or* **does not exist**

2, 4, 8 ($f'=0$), 6 (f' d.n.e.)

(b) On what intervals is f increasing or decreasing?

where the derivative is **positive** or ***negative***

$(0,2)$, $(4,6)$, $(8,\infty)$ increasing; $(2,4)$, $(6,8)$ decreasing

(c) Where does f have a local maximum or minimum?

where the derivative **changes sign**

At $x=2, 6$ a local max, f' sign changes from + to −

At $x=4, 8$ a local min, f' sign changes from − to +

Example A. For the function $f(x) = x^2(x^2-4x+4)$ $= x^4 - 4x^3 + 4x^2$

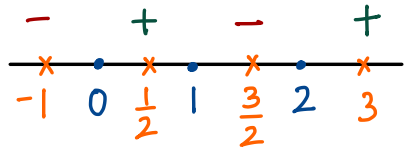
(a) Compute $f'(x)$. $f'(x) = 4x^3 - 12x^2 + 8x$

$= 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$

(b) Find the critical numbers of $f'(x)$.

$f'(x) = 4x(x-1)(x-2) = 0 \rightsquigarrow x = 0, 1, 2$

(c) Find the intervals on which f is increasing or decreasing.



$f'(-1) = -24 < 0$

$f'(1/2) = \frac{3}{2} > 0$

$f'(3/2) = -\frac{3}{2} < 0$

$f'(3) = 24 > 0$

$f' > 0$ on $(0,1)$ and $(2,\infty)$

f is increasing on $(0,1)$ and $(2,\infty)$

$f' < 0$ on $(-\infty, 0)$ and $(1,2)$

f is decreasing on $(-\infty, 0)$ and $(1,2)$

(d) Find the local maximum and minimum values of f.

Looking at the sign chart for f' in (c), using the 1st derivative test

Local max at $x = 1$ (sign changes from + to −): $f(1) = 1$

Local min at $x = 0, 2$ (sign changes from − to +): $f(0) = f(2) = 0$

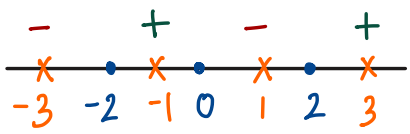
Problem 3. For the function $f(x) = x^4 - 8x^2 + 3$.

(a) Compute $f'(x)$. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$

(b) Find the critical numbers of $f'(x)$.

$f'(x) = 4x(x-2)(x+2) = 0 \rightsquigarrow x = 0$ or $x = -2$ or $x = 2$

(c) Find the intervals on which f is increasing or decreasing.



$f'(-3) = -60 < 0$ $f'(1) = -12 < 0$

$f'(-1) = 12 > 0$ $f'(3) = 60 > 0$

$f' > 0$ on $(-2, 0)$ and $(2, \infty)$

f is increasing on $(-2, 0)$ and $(2, \infty)$

$f' < 0$ on $(-\infty, -2)$ and $(0, 2)$

f is decreasing on $(-\infty, -2)$ and $(0, 2)$

(d) Find the local maximum and minimum values of f.

Looking at the sign chart for f' in (c), using the 1st derivative test

Local max at $x = 0$ (sign changes from + to −): $f(0) = 3$

Local min at $x = \pm 2$ (sign changes from − to +): $f(2) = f(-2) = -13$

The second derivative f'' of a function $f(x)$ also tells us quite a bit about the shape of its graph.

Concavity Test. The second derivative of a function $f(x)$ tells us the **curvature** of the function.

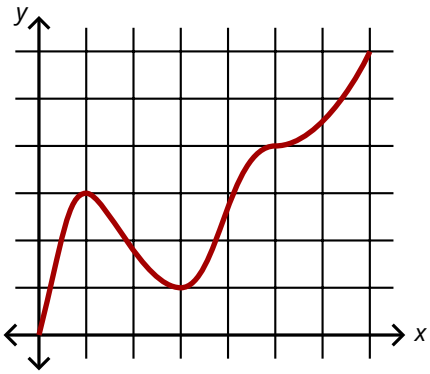
- If $f''(x) > 0$ on an interval, then f is concave upwards on that interval.
- If $f''(x) < 0$ on an interval, then f is concave downwards on that interval.

cup

frown

Inflection Point. We call a number p an **inflection point** of $f(x)$ if it changes concavity at p

Problem 4. This is the graph from **Problem 1**.



(a) Estimate the open intervals where f is concave upward.

$(2,4)$ and $(5,7)$

(b) Estimate the open intervals where f is concave downward.

$(0,2)$ and $(4,5)$

(c) Determine points of inflection.

$x=2$, $x=4$, $x=5$

Connecting First and Second Derivatives. The second derivative f'' is the *first derivative* of f'. So,

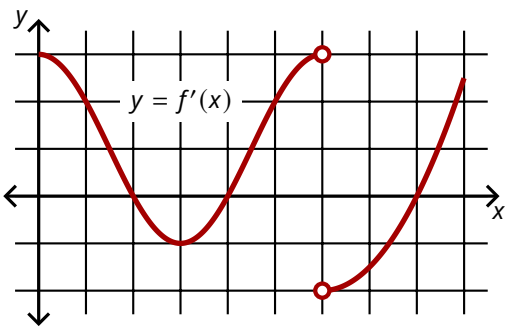
- If $f''(x) > 0$ on an interval, then f' is increasing on that interval.
- If $f''(x) < 0$ on an interval, then f' is decreasing on that interval.

An **inflection point** p of $f(x)$ is that critical no. of f' where f' inc/dec and then inc/dec from *left of p to right of p.*

inc = increasing ;

dec = decreasing

Problem 5. This is the graph from **Problem 2**, a graph of $f'(x)$ of some function f.



(a) On what intervals is f concave upward or downward?

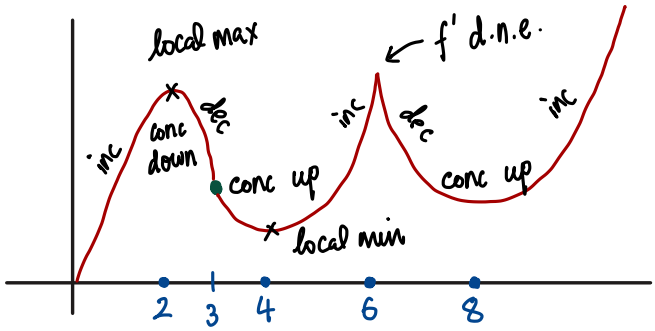
f' is increasing for $(3,6)$ & $(6,9)$ so on this interval $f''>0$ and f is concave up

f' is decreasing for $(0,3)$ so on this interval $f''<0$ and f is concave down.

(b) Where does f have a point of inflection?

At $x=3$, f' goes from dec to inc, i.e. $f''<0$ to $f''>0$, i.e., f goes from concave up to concave down.

(c) Using (a) and (b) of this problem and Problem 2 (a) & (b), sketch a graph of f assuming $f(0) = 0$.



One possible answer!

Second Derivative Test for Local Max/Min. Let f'' be continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum **at c.**
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum **at c.**
- If $f''(c) = 0$, then test is inconclusive

Example B. Consider the function $f(x) = x^4 - 8x^2 + 3$ from **Problem 3**.

(a) Use the second derivative test to verify that the points found in Example A are indeed where f has a local maxima or minima.

Critical numbers s.t. $f' = 0$: $x = 0$, $x = 2$, $x = -2$

$f''(x) = 12x^2 - 16$

$= 12(x^2 - 16/12)$

$= 12(x^2 - 4/3)$

$f''(0) = -16 < 0$ local max at $x=0$

$f''(2) = 32 > 0$

$f''(-2) = 32 > 0$

local min at $x = \pm 2$

Exactly what we got in Prob. 3

(b) Compute the local maxima and minima.

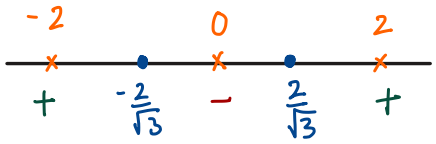
$f(0) = 3 \leftarrow$ local max $\qquad f(2) = f(-2) = -13 \leftarrow$ local min

(c) Find the intervals of concavity and the inflection points.

$f''(x) = 12(x^2 - 2/3) = 0$

$\rightsquigarrow x = 2/\sqrt{3}$ or $x = -2/\sqrt{3}$

$\approx 1.155 \qquad \approx -1.155$



$f''(-1) = f''(1) = 32 > 0, \quad f''(0) = -16 < 0$

f is concave upwards on $(-\infty, -\frac{2}{\sqrt{3}})$ & $(\frac{2}{\sqrt{3}}, \infty)$

f is concave downwards on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$

Based on sign chart, inflection points are $x = \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$

Problem 6. Consider the function $f(x) = x^4 - 4x^3 + 4x^2$ from **Example A**.

(a) Use the second derivative test to verify that the points found in Example A are indeed where f has a local maxima or minima.

Critical numbers s.t. $f' = 0$: $x = 0, \; x = 1, \; x = 2$

$f''(x) = 12x^2 - 24x + 8$

$= 4(3x^2 - 6x + 2)$

$f''(0) = 8 > 0$ local min at $x = 0$

$f''(1) = -4 < 0$ local max at $x = 1$

$f''(2) = 8 > 0$ local min at $x = 2$

Exactly what we got in Example A

(b) Compute the local maxima and minima.

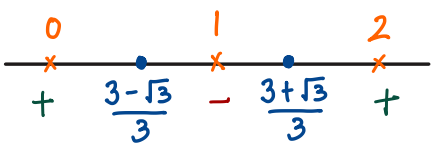
$f(1) = 1 \leftarrow$ local max $\qquad f(0) = f(2) = 0 \leftarrow$ local min

(c) Find the intervals of concavity and the inflection points.

$f''(x) = 4(3x^2 - 6x + 2) = 0$

$\rightsquigarrow x = \frac{3+\sqrt{3}}{3}$ or $x = \frac{3-\sqrt{3}}{3}$

$\approx 1.58 \qquad \approx 0.42$



$f''(0) = f''(2) = 8 > 0, \quad f''(1) = -4 < 0$

f is concave upwards on $(-\infty, \frac{3-\sqrt{3}}{3})$ & $(\frac{3+\sqrt{3}}{3}, \infty)$

f is concave downwards on $(\frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3})$

Based on sign chart, inflection points are $x = \frac{3-\sqrt{3}}{3}, \frac{3+\sqrt{3}}{3}$

Recap Problems

The aim is to review Day 21 material; so have the filled-in Day 21 notes in front of you as a reference!

Recap Problem 1. Compute the critical numbers of $f(x) = x^{4/3} - 4x^{1/3}$. You should get *two critical numbers*.

Recap Problem 2. Compute the absolute maxima and minima of the function $f(x) = x^{4/3} - 4x^{1/3}$ in the interval $[-1, 4]$.

Recall. *You want to compute the function output at critical numbers in the interval* $[-1, 4]$ *and the output at the end points of the interval.*

Name: Deewang

Day 25. Optimisation

(based on Sections 4.6)

- Translate a practical word problem into a mathematical optimisation problem.
- Identify the objective function (to be maximised/minimised) and the constraint equation (the given condition).

Main Idea.

By **optimisation** we mean "finding the best possible outcome" – *the most area, the least cost, the fastest time*, etc.
For this, we wish to find the absolute maximum or minimum of some function.

Therefore the mathematical input is our ability to determine local max/min which we may do as follows for a continuous function $f(x)$ defined on an interval $[a, b]$.

Determining Absolute Max/Min on $[a,b]$.

Step 1. Determine critical numbers by computing those numbers c such that $f'(c) = 0$.

Step 2. Discard any critical numbers that are not in the interval $[a, b]$.

Step 3. To determine if the remaining critical numbers are local max/min, use either of the following tests:

- **First Derivative Test.** Take two test values really close to c, one on the left and one on the right and compute the value of f' on these two test values.

  If the sign of f' changes from positive to negative from left of c to right of c, then there is a *local max* at c.

  If the sign of f' changes from negative to positive from left of c to right of c, then there is a *local min* at c.

- **Second Derivative Test.** If $f''(c) = 0$, the test is inconclusive.

  If $f''(c) < 0$ (concave down), then there is a *local max* at c.

  If $f''(c) > 0$ (concave down), then there is a *local min* at c.

Step 4. Compute $f(c)$ for all the c's you have determined to be local max/min, and also compute $f(a)$ and $f(b)$. The *largest of these numbers is the absolute max*, and *the smallest of these numbers is the absolute min*.

Example A. What is the maximum vertical distance between $y = x + 2$ and $y = x^2$ for $-1 \le x \le 2$?

Main Question: *What is the objective function, i.e., function we wish to maximise (or minimise)?*

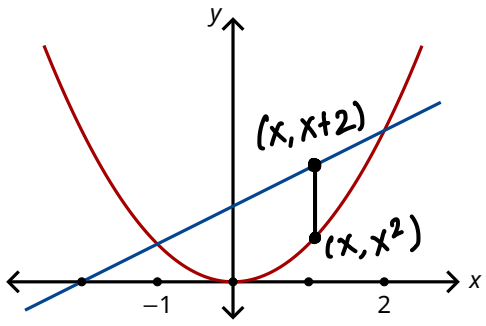
$D(x) = x+2-x^2 \rightsquigarrow D'(x) = 1-2x$

$1-2x = D'(x) = 0 \rightsquigarrow x = 1/2$

$D(-1) = 0, \quad D(2) = 0, \quad D(1/2) = 2.25$

Maximum vertical distance = 2.25

at $x = 0.5$

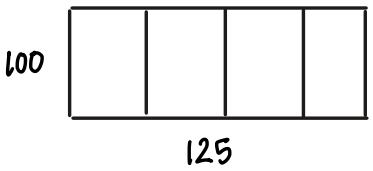


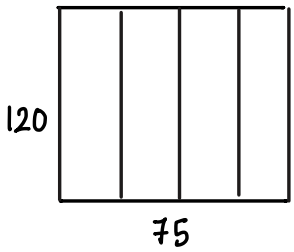
Problem-Solving Strategy for Optimisation Problems

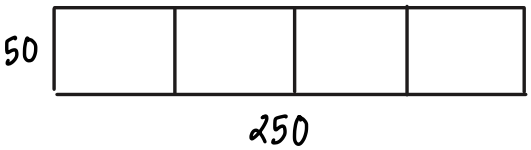
1. **Understand the Problem.** Read the problem carefully until it is clearly understood. Identify the unknown quantities, the given quantities, and the given conditions. Specifically, determine the *objective* (the quantity to be maximised or minimised) and the *constraint* (the condition limiting the objective).
2. **Draw a Diagram.** In most problems, it is useful to draw a diagram. Label the diagram with the given and required quantities, *assigning variables to the unknown values*.
3. **Introduce notation.** Assign a symbol (like A if the problem is about area) to the objective function. Write an equation for the objective function in terms of the variables from your diagram (e.g., $A = \ell \cdot w$). Then, write one or more constraint equations that relate your variables based on the problem's given conditions (e.g., $2\ell + 2w = 100$ if the constraint is about the perimeter).
4. **Create a Function of One Variable.** Use the constraint equation(s) to solve for one variable and substitute it into the objective function. This simplifies the objective to a function of a single independent variable, such as $A(x)$.
5. **Determine the Feasible Domain.** Once you have your single-variable function $A(x)$, determine the domain of x-values that are physically possible in the context of the problem (e.g., lengths must be non-negative). If possible, express this as a closed interval $[a, b]$.
6. **Find the Absolute Maximmum or Minimum.** Use calculus techniques given in the previous page to find the absolute max/min of the function $A(x)$ on $[a, b]$.
7. **Answer the Question.** *Re-read the problem one last time* to ensure you are answering the specific question being asked (e.g., finding the dimensions, the maximum volume, etc.).

Example B. We have 750 ft of fencing and wish to enclose a rectangular area that is divided up into four pens with fencing parallel to one side of the rectangle. What is the largest possible area of these four pens?

(a) Draw three diagrams: one with wide and shallow pens, one with deep and narrow pens, and one in between. Pick some numbers that realise these diagrams and find the total areas of the pens for these three situations.



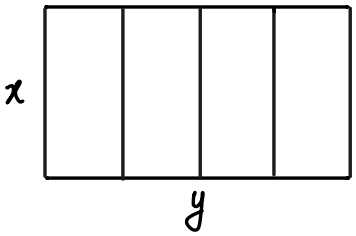




(b) Draw a "generic" diagram, introduce notation and label the diagram with this notation.

x denote the length of the two sides and three dividers

y denote the length of the other two sides

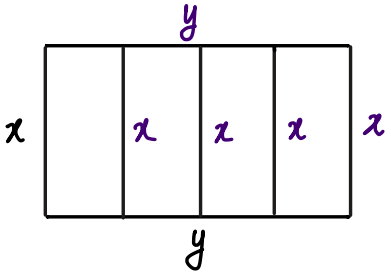


(c) Write down an expression for the total area enclosed, and write down an expression for the total amount of fencing used.

Area A = length × width $= yx$

Length of fencing = 750

$\rightsquigarrow 5x + 2y = 750$



(d) Use (c) to create a function giving the area in terms of one variable only.

$5x + 2y = 750 \rightsquigarrow y = \frac{750 - 5x}{2} = 375 - \frac{5}{2}x$

$A(x) = yx = \left(375 - \frac{5}{2}x\right) \cdot x = 375x - \frac{5}{2}x^2$

(e) Solve the problem and compare your answer with your estimate from (a).

$A'(x) = 375 - 5x = 0 \rightsquigarrow x = 75$

$A''(x) = -5 < 0$ there is an absolute max when $x = 75$

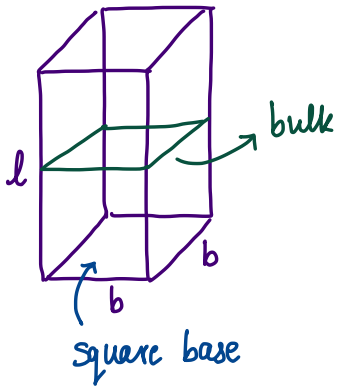
Then $y = 375 - \frac{5}{2} \cdot 75 = 187.5$ Largest area $= yx = 187.5 \times 75$

$= 14,062.5 \text{ ft}^2$

Problem 1. A package mailed by USPS may not measure more than 108 inches in height plus bulk. If a package placed on its base, height is the dimension perpendicular to the base and bulk is the total distance around the package in a direction perpendicular to height. Find the dimensions of the largest volume rectangular box with a square base that is possible to mail via USPS.

bulk $= 4b$ $\quad 108 = \ell + 4b$, $\ell, b > 0$

Volume = length × base × base $= \ell b^2$



$\ell = 108 - 4b$

$V(b) = (108 - 4b)b^2 = 108b^2 - 4b^3$

$V'(b) = 216b - 12b^2 = 12b(18 - b) = 0 \rightsquigarrow b = 18$ (since $b \neq 0$)

$V''(b) = 216 - 24b \rightsquigarrow V''(18) = -216 < 0$ Max at $b = 18$ in

$\ell = 108 - 4b = 36$ in $\rightsquigarrow$ Largest volume $V = 36 \cdot 18^2 = 11,664 \text{ in}^3$

Problem 2. A right circular cylinder is to be designed to hold a liter of water. Find the dimensions of such a cylinder which uses the least amount of material in its construction.

For a right circular cylinder with radius r and height h, the volume is $\pi r^2 h$ and surface area is $2\pi rh + 2(\pi r^2)$.

We want to minimise surface area $S = 2\pi rh + 2\pi r^2$

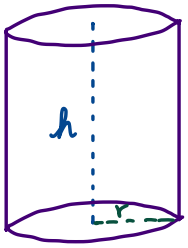
Volume $= \pi r^2 h = 1\ell = 1000\ cm^3 \rightsquigarrow h = 1000/\pi r^2$

$S(r) = 2\pi r \cdot 1000/\pi r^2 + 2\pi r^2 = 2000/r + 2\pi r^2$

$S'(r) = -2000/r^2 + 4\pi r = 0 \rightsquigarrow 4\pi r^2 = 2000/r$

$\rightsquigarrow r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42\ cm \rightsquigarrow S''(r) = 4000/r^3 + 4\pi r \rightsquigarrow S''(5.42) > 0$

Surface area minimised at $r = 5.42\ cm \rightsquigarrow h = 1000/\pi r^2 \approx 10.84\ cm$



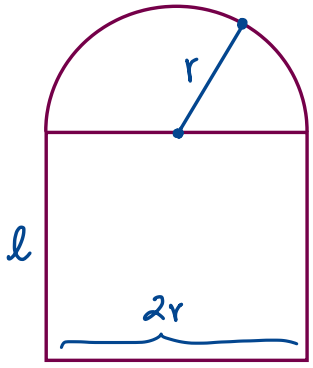
h is height
r is radius

Problem 3. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of the window of maximum area if the total outer perimeter is 18 meters.

We want to maximise

area $A = \frac{1}{2}\pi r^2 + \ell \cdot 2r$

base rectangle has width = diameter of the top circle



Outer perimeter $= \frac{1}{2}(2\pi r) + \ell + 2r + \ell = 18\ m$

$\rightsquigarrow 2\ell = 18 - \pi r - 2r \rightsquigarrow \ell = \frac{18 - \pi r - 2r}{2}$

ℓ is the length
r is the radius

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$$A(r) = \frac{1}{2}\pi r^2 + 2\ell r = \frac{1}{2}\pi r^2 + (18 - \pi r - 2r)\cdot r$$

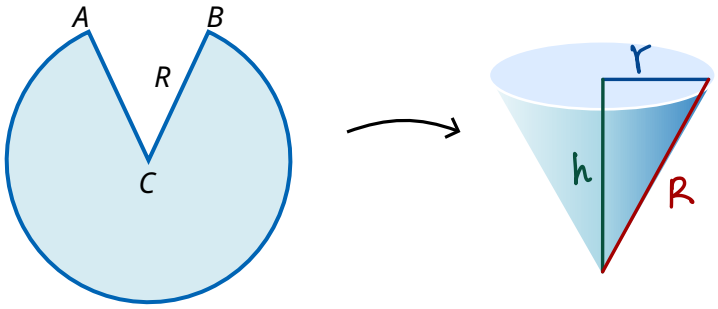
$$= 18r - 2r^2 - \frac{1}{2}\pi r^2$$

$$A'(r) = 18 - 4r - \pi r = 0 \rightsquigarrow r = 18/(4+\pi) \approx 2.52\text{m}$$

$$A''(r) = -4 - \pi < 0 \rightsquigarrow \ell = \frac{18 - \pi r - 2r}{2} = \frac{18}{4+\pi} \approx 2.52\text{m}$$

$r = 18/(4+\pi)$ maximises!

Problem 4. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a paper of radius R by cutting out a sector and joining the edges cup.



$$r^2 + h^2 = R^2$$

$$\rightsquigarrow r^2 = R^2 - h^2$$

Volume $V = \frac{1}{3}\pi r^2 h$

$$V(h) = \frac{1}{3}\pi(R^2 - h^2)\cdot h = \frac{\pi}{3}(R^2 h - h^3)$$

$$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0 \rightsquigarrow h^2 = R^2/3 \rightsquigarrow h = R/\sqrt{3}$$

$$V''(h) = -2\pi h \rightsquigarrow V''(R/\sqrt{3}) = -2\pi\cdot R/\sqrt{3} < 0$$

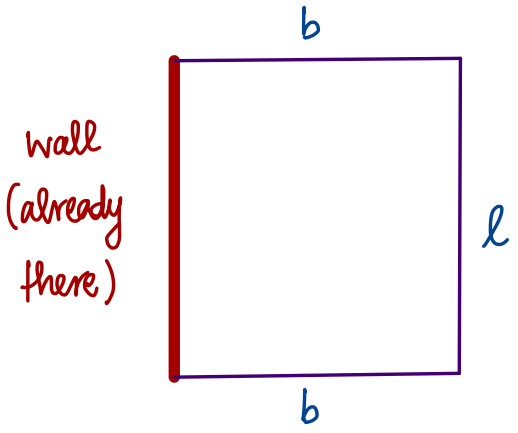
$h = R/\sqrt{3}$ maximises volume

$$\text{Max volume} = V\left(\frac{R}{\sqrt{3}}\right) = \frac{\pi}{3}\left(R^2\cdot R/\sqrt{3} - (R/\sqrt{3})^3\right) = \frac{2}{9\sqrt{3}}\pi R^3$$

Problem 5. A company is building a new warehouse and needs to fence off a rectangular storage yard beside it. The warehouse wall will serve as one side of the rectangle, so fencing is only needed for the other three sides. The storage yard must have a total area of 1,200 m^2.

The fencing used for the two sides perpendicular to the warehouse wall is standard and costs \$20 per metre. The fencing for the side parallel to the warehouse (the "front" gate) needs to be more secure and costs \$30 per metre.

What dimensions (length and width) should the storage yard have to minimise the total cost of the fencing?



\$30/m

Perimeter of fencing $= \ell + 2b$ → \$20/m

Total Cost $= 30\ell + 40b$

Area $= \ell b = 1200 \rightsquigarrow b = \frac{1200}{\ell}$

$$C(\ell) = 30\ell + \frac{48000}{\ell} \rightsquigarrow C'(\ell) = 30 - \frac{48000}{\ell^2} = 0$$

$$\rightsquigarrow \frac{48000}{\ell^2} = 30 \rightsquigarrow \ell^2 = 1600$$

$$\rightsquigarrow \ell = 40$$

$$C''(\ell) = \frac{96000}{\ell^3} \rightsquigarrow C''(40) = \frac{96000}{40^3} > 0$$

$\ell = 40$ ✪ minimises cost

$$b = \frac{1200}{40} = 30$$ ✪

Recap Problems

The aim is to review Day 24 material; so have the filled-in Day 24 notes in front of you as a reference!

Recap Problem 1. For the function $f(x) = \frac{x^5}{5} - \frac{5x^4}{4} + 2x^3$, find all

(a) critical numbers

(b) intervals where the function is increasing and decreasing

(c) intervals where the function is concave upwards and downwards

(d) inflection points

(e) local maxima and minima

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Name: Deewang

Day 26. Curve Sketching

(based on Sections 4.5, 4.6)

- Identify features of a function using the first and second derivative
- Use the information to sketch the graph of the function

Example A. Sketch the graph of a function that satisfies all of the following criteria:

(A) $f'(0) = f'(2) = f'(4) = 0$ horizontal tangent lines at $x = 0, 2, 4$

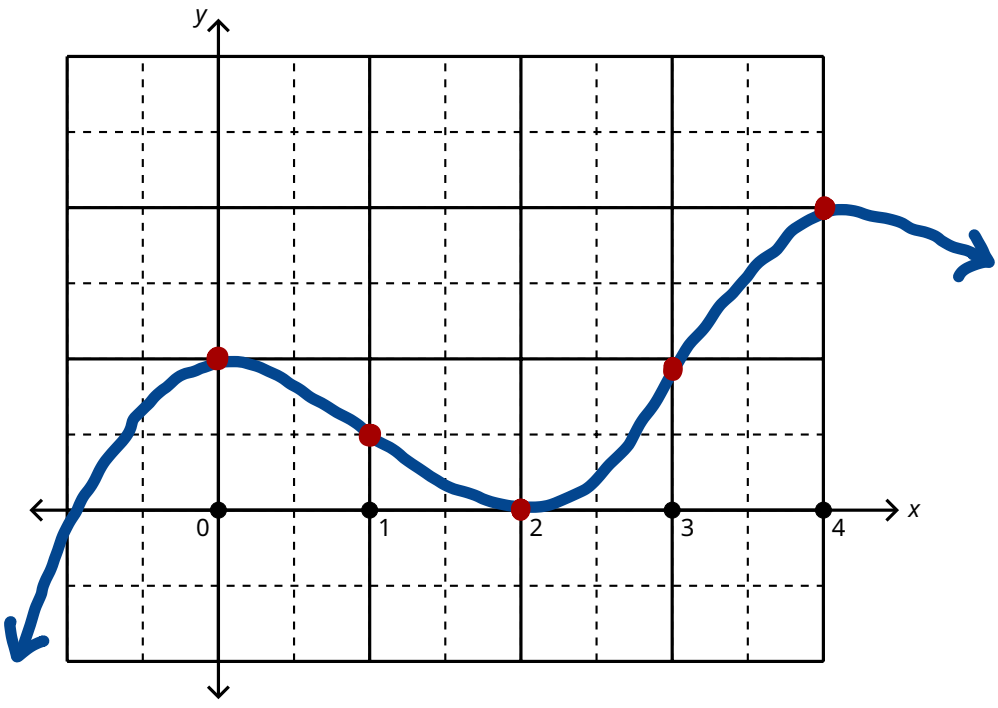
(B) $f'(x) > 0$ if $x < 0$ or $2 < x < 4$ increasing on $(-\infty, 0)$ & $(2, 4)$

(C) $f'(x) < 0$ if $0 < x < 2$ or $x > 4$ decreasing on $(0, 2)$ & $(4, \infty)$

(D) $f''(x) > 0$ if $1 < x < 3$ concave upwards on $(1, 3)$

Inflection points at $x = 1$ & 3

(E) $f''(x) < 0$ if $x < 1$ or $x > 3$ concave downwards on $(-\infty, 1)$ & $(3, \infty)$



Example B. Consider the function $f(x) = \dfrac{x^2 + 5x}{25 - x^2}$. Determine the following.

(a) **Domain**

(b) **Intercepts**. x-intercepts and y-intercepts

(c) **Asymptotes**. vertical and horizontal

(d) **Intervals of Increase or Decrease**

(e) **Local Maximum or Minimum**

(f) **Concavity and Points of Inflection**

(g) **Sketch the graph**

(a) domain : $25 - x^2 \neq 0 \rightsquigarrow x \neq \pm 5$

(b)

y-intercept : value of $f(0) = \dfrac{0^2 + 5\cdot 0}{25 - 0^2} = 0$ y-intercept is $(0,0)$

x-intercept : solve $f(x) = 0 \rightsquigarrow \dfrac{x^2+5x}{25-x^2} = 0 \rightsquigarrow x^2 + 5x = 0$, $x(x+5) = 0$

x-intercept is $(0,0)$ $\rightsquigarrow x = 0$ & -5 ↖ not in the domain

(c) $f(x) = \dfrac{x^2+5x}{25-x^2} = \dfrac{x(x+5)}{(5+x)(5-x)} = \dfrac{x}{5-x}$ when $x \neq -5$

Vertical Asymptote

$5 - x = 0 \rightsquigarrow \boxed{x = 5}$

Horizontal Asymp : $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{x}{5-x} = \lim\limits_{x \to \pm\infty} \dfrac{1}{\frac{5}{x} - 1}$

$\boxed{y = -1}$

$= \dfrac{1}{0-1} = -1$

(d) $f(x) = \frac{x}{5-x}$ when $x \neq -5$, $f'(x) = \frac{(5-x) - x(-1)}{(5-x)^2} = \frac{5}{(5-x)^2}$

always positive

f is increasing $(-\infty, \infty)$

(e) $f'(x) = \frac{5}{(5-x)^2}$ is not defined when $(5-x)^2 = 0 \rightsquigarrow x = 5$

But $x = 5$ is not in the domain. No local max/min

(f) $f''(x) = \frac{d}{dx}\left(5 \cdot (5-x)^{-2}\right) = 5 \cdot (-2) \cdot (5-x)^{-3} \cdot (-1)$

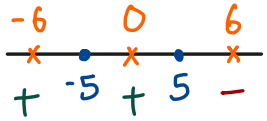
$= \frac{10}{(5-x)^3} \neq 0$ for x in the domain

Concave up on $(-\infty, -5)$ & $(-5, 5)$

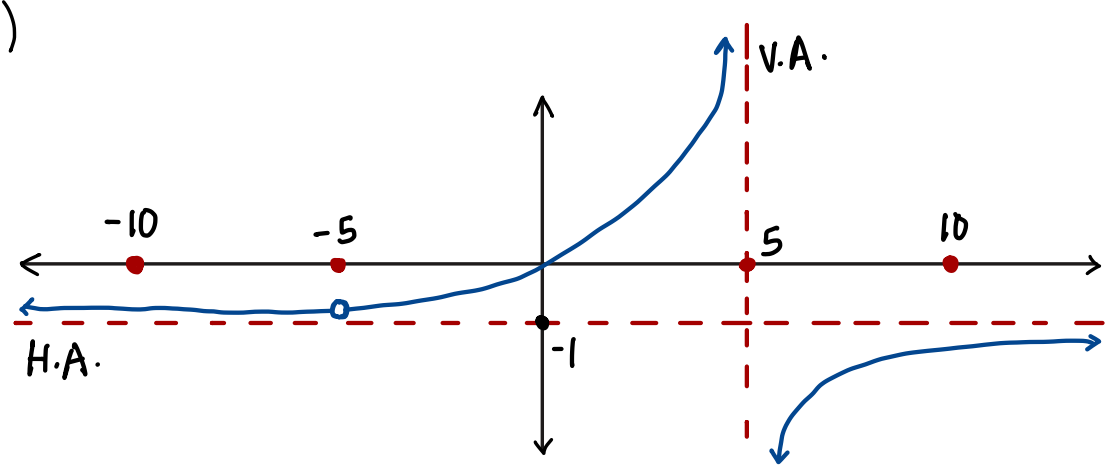
Concave down $(5, \infty)$

f'' is undefined for $x = 5 \rightsquigarrow$ (not in domain)

Sign of f''



(g)



Problem 1. Sketch the graph of a function that satisfies all of the following criteria:

(A) $f'(-1) = f'(1) = f'(3) = 0$ horizontal tangent lines at $x = -1, 1, 3$

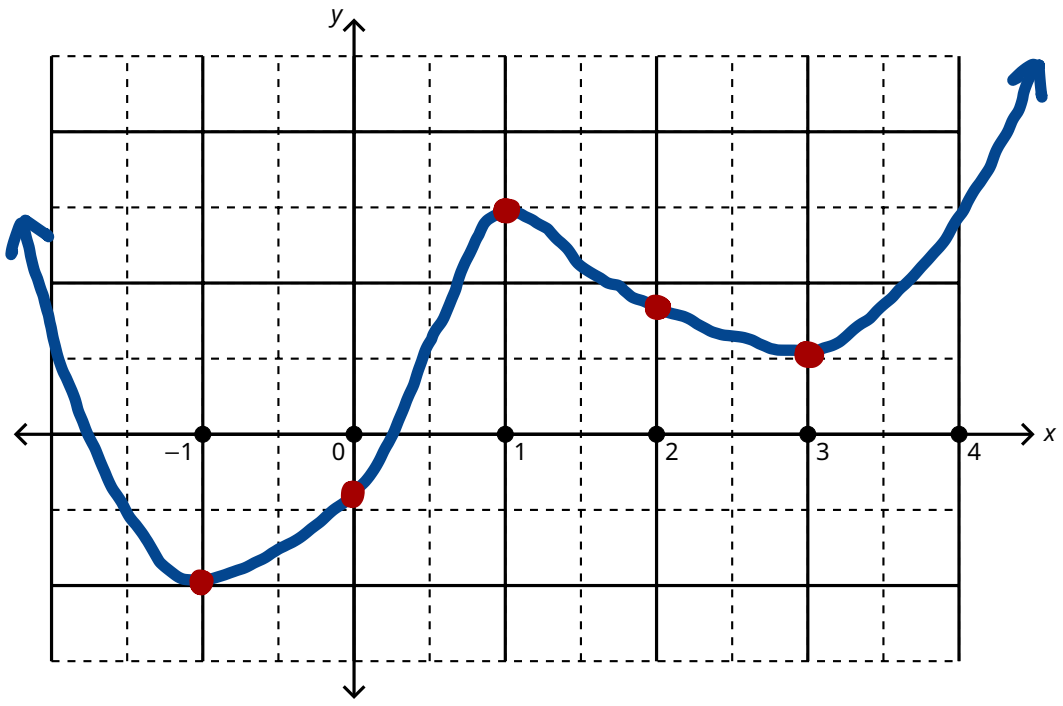
(B) $f'(x) < 0$ if $x < -1$ or $1 < x < 3$ decreasing on $(-\infty, -1)$ & $(1, 3)$

(C) $f'(x) > 0$ if $-1 < x < 1$ or $x > 3$ increasing on $(-1, 1)$ & $(3, \infty)$

(D) $f''(x) < 0$ if $0 < x < 2$ concave downwards on $(0, 2)$

Inflection points at $x = 0$ & 2

(E) $f''(x) > 0$ if $x < 0$ or $x > 2$ concave upwards on $(-\infty, 0)$ & $(2, \infty)$



Problem 2. For the function $f(x) = x^4 - 2x^3$.

(a) **Domain**

(b) **Intercepts**. x-intercepts and y-intercepts

(c) **Intervals of Increase or Decrease**

(d) **Local Maximum or Minimum**

(e) **Concavity and Points of Inflection**

(f) **Sketch the graph**

(a) domain : $(-\infty, \infty)$

(b) y-intercept : value of $f(0) = 0$

y-intercept is $(0,0)$

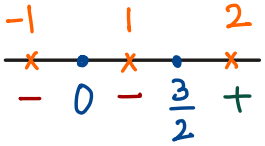
x-intercept : solve $f(x) = 0 \rightsquigarrow x^4 - 2x^3 = 0$

$\rightsquigarrow x^3(x-2) = 0 \rightsquigarrow x = 0$ & $x = 2$

x-intercepts are $(0,0)$ & $(2,0)$

(c) $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3) = 0 \rightsquigarrow x = 0, 3/2$

Sign of f'



$f'(-1) = -10 < 0$

$f'(1) = -2 < 0$

$f'(2) = 8 > 0$

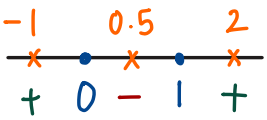
f is dec on $(-\infty, 3/2)$

f is inc on $(3/2, \infty)$

(d) From (c), using the 1st derivative test local min at $x = 3/2$

(e) $f''(x) = 12x^2 - 12x = 12x(x-1) = 0 \rightsquigarrow x = 0, 1$

Sign of f''



$f'(-1) = 24 > 0$

$f'(0.5) = -3 < 0$

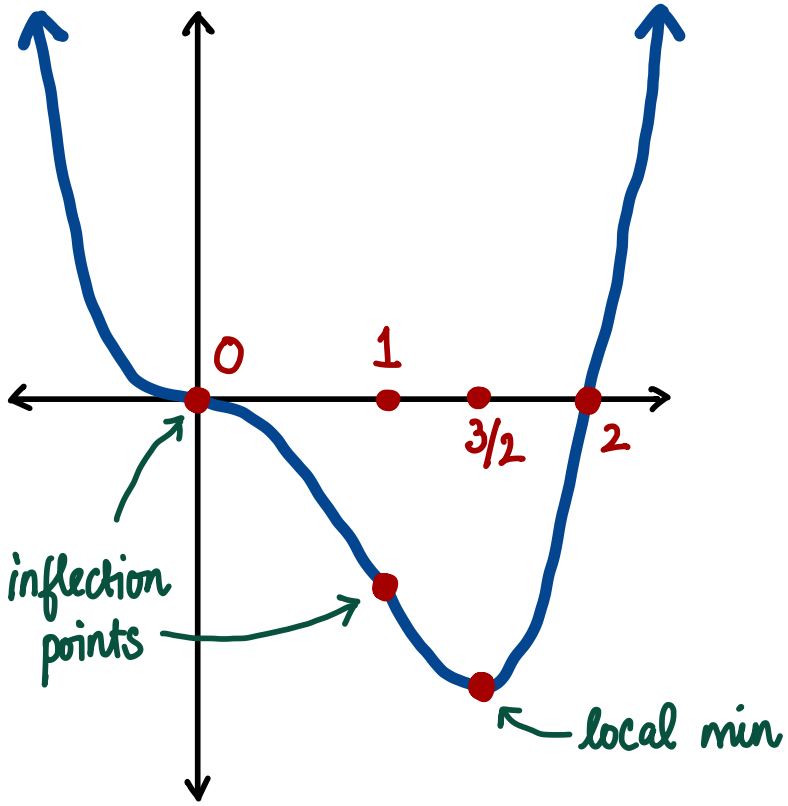
$f'(2) = 24 > 0$

f is concave down on $(0,1)$

f is concave up on $(-\infty, 0)$ and $(1, \infty)$

$x = 0, 1$ are inflection points

(f)



Recap Problems

The aim is to review Day 25 material; so have the filled-in Day 25 notes in front of you as a reference!

Recap Problem 1. If 3 m^2 is available to make a box with a **square base** and an **open top**, find the largest volume possible of the box. This means the answer needs to be the volume value.

Volume of a box with length ℓ, width w and height h is ℓwh. The surface area of a box is the sum of areas of each side which are rectangles, so if all sides are to be counted (including the top) the surface area of a box is $2\ell w + 2wh + 2\ell h$.

We reiterate that the box will have a **square base** and an **open top**.

Algebra Skills Lab 1

Powers & Algebra of the "Difference Quotient"

Manipulating Powers

In our class, with our eventual aim being to "differentiate", we rarely leave expressions, if they can be simplified, as fractions or radicals. To take a derivative easily, you usually need to rewrite everything in the form x^r.

The Rewrite Rules.

I. Roots are Fractions: In Calculus, we want to avoid radical signs like $\sqrt{x}$, we always rewrite:

$$\sqrt[n]{x^m} = x^{m/n}$$

That is, the number in the root goes in the denominator of the exponent. For example,

$$\sqrt{x} = x^{1/2}, \qquad \sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$$

II. Reciprocals are Negative: If x is downstairs, move it upstairs with a negative in the exponent:

$$\frac{1}{x^n} = x^{-n}$$

For example,

$$\frac{1}{x} = x^{-1}, \qquad \frac{3}{x^2} = 3x^{-2}$$

III. Beware "Forbidden Move":

- $(x+y)^2 \neq x^2 + y^2$ (you must **always FOIL** or expand)
- $\sqrt{x^2+y^2} \neq x + y$ (square roots **do not distribute** over addition)

Drill A. Rewrite the following expressions as a sum of terms in the form $c \cdot x^n$.

Do not leave any x's in the denominator or under a radical.

(1) $\dfrac{7}{x^5}$

(2) $\dfrac{4}{\sqrt[4]{x^3}}$

(3) $\sqrt[4]{t} \cdot t^5$ (add exponents)

(4) $\sqrt{4t} \cdot t^3$

(5) $\dfrac{y^2+5y-2}{y}$ (split the fraction)

(6) $\dfrac{y-1-3y^2}{\sqrt{y}}$

Drill B. Rewrite the following expressions as a fraction or/and with radicals.

(1) $-2x^{-4}$

(2) $x^{-1/2}$

(3) $5x^{-4/3}$

(4) $x^{-(-2)}$

Function Notation & The "Difference Quotient"

Revisiting Function Notation.
Given a function, for example,

$$f(x) = x^2 - 3x,$$

one should think of x as an *empty slot*:

$$f(\quad) = (\quad)^2 - 3 \cdot (\quad).$$

I. Constant Input: If you wish to compute $f(4)$, it means 4 goes into the empty slot:

$$f(\,4\,) = (\,4\,)^2 - 3 \cdot (\,4\,) = 16 - 12 = 4$$

II. Variable Input: Similarly, $f(x+h)$ means "everywhere you see an x, replace it with $x+h$"

$$f(\,x+h\,) = (\,x+h\,)^2 - 3 \cdot (\,x+h\,)$$

III. Beware "Forbidden Move": $f(x+h) \neq f(x) + h$

Drill C. For $f(x)$ below, what is $f(x+h)$?

(1) $f(x) = x(x+2)$

(2) $f(x) = \sqrt{x+1}$

(3) $f(x) = \sin(x^2)$

(4) $f(x) = \dfrac{x}{x+3}$

(5) $f(x) = 2^{2x}$

Drill D. For functions $f(x)$ below, calculate the **Difference Quotient:**

$$\frac{f(x+h) - f(x)}{h}$$

by following these steps:

Step 1. Write out $f(x+h)$ expanded (FOIL).

Step 2. Subtract the original $f(x)$ from it. (Notice terms should cancel!).

Step 3. Divide by h (Factor an h out of the top first).

(1) $f(x) = x^2 - 3x$

(2) $f(x) = x(x+2)$

Algebra Skills Lab 2

The Algebra of Limits: Solving the 0/0 Problem

Direct substitution often leads to the indeterminate form $0/0$. This is not a dead end; it may imply that one can use algebra to simplify the expression. Today, we practice the three specific techniques used to "fix" these broken limits.

Rationalising (The Conjugate Method)

When you see a square root involved in a sum or difference, e.g., $\sqrt{x}-3$, you generally cannot factor it easily. Instead, we use the **conjugate**.

Strategy: The Conjugate. Recall that $(A-B)(A+B)=A^2-B^2$. We eliminate square roots this way: either A or B will be our expression with a square root and we utilise the fact that *squaring a square-root gets rid of the square-root*.

I. If you have $\sqrt{\text{stuff}}-\text{number}$, multiply and divide by $\sqrt{\text{stuff}}+\text{number}$, its **conjugate**

$$(\text{expression with } \sqrt{\text{stuff}}-\text{number}) = (\text{expression with } \sqrt{\text{stuff}}-\text{number}) \cdot \frac{\sqrt{\text{stuff}}+\text{number}}{\sqrt{\text{stuff}}+\text{number}}$$

For example,

$$\frac{\sqrt{1+h}-1}{h} = \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}, \qquad \frac{x}{\sqrt{x+2}-4} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

II. One uses $(\sqrt{\text{stuff}}-\text{number})(\sqrt{\text{stuff}}+\text{number}) = (\sqrt{\text{stuff}})^2-(\text{number})^2 = \text{stuff}-(\text{number})^2$

$$(\sqrt{1+h}-1)(\sqrt{1+h}+1) = (\sqrt{1+h})^2-1^2 = (1+h)-1$$

$$(\sqrt{x+2}-4)(\sqrt{x+4}+2) = (\sqrt{x+4})^2-2^2 = (x+4)-4$$

III. **Crucial Advice:** Multiply out the side with the roots (to clear them), but **DO NOT** multiply out the other side. Leave it factored so you can cancel the problematic term later. For example,

$$\frac{\sqrt{1+h}-1}{h} = \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \frac{(1+h)-1}{h(\sqrt{1+h}-1)}$$

$$\frac{x}{\sqrt{x+4}-2} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{x(\sqrt{x+4}+2)}{(x+4)-4}$$

In the first example, we multiplied out the numerator since it is being multiplied to its conjugate, but kept the *denominator as is*. Similarly, in the second example we multiplied out the denominator since it is being multiplied to its conjugate, but kept the *numerator as is*.

Drill A. Simplify the following expressions by rationalising the numerator. Your final goal is to cancel out the denominator.

(1) $\dfrac{\sqrt{x+9}-3}{x}$ (2) $\dfrac{\sqrt{1-h}-1}{h}$ (3) $\dfrac{2-\sqrt{4-x}}{x}$

Complex Fractions (Fractions within Fractions)

Limits often result in "double-decker" fractions. The best approach is to separate the division from the subtraction.

Strategy: Rewrite Division as Multiplication. Handling a fraction of the form $\dfrac{\text{Fraction}}{\text{Denominator}}$ can be confusing.

I. Pull out the Denominator: Rewrite division by D as multiplication by $\dfrac{1}{D}$.

$$\frac{\text{Complex Fraction}}{D} = \frac{1}{D} \cdot (\text{Complex Fraction}) \quad \rightsquigarrow \quad \text{for example, } \frac{\frac{1}{x} - 1}{x - 1} = \frac{1}{x - 1} \cdot \left(\frac{1}{x} - 1\right)$$

II. Simplify the Bracket: Now focus solely on combining the fractions inside the bracket.

$$\frac{1}{x - 1} \cdot \left(\frac{1}{x} - 1\right) = \frac{1}{x - 1} \cdot \left(\frac{1 - x}{x}\right) = \frac{1}{x} \cdot \frac{1 - x}{x - 1}$$

III. Watch for the "Reverse" Factor: You will often end up with $(a - x)$ on top and $(x - a)$ on the bottom.

Recall: $a - x = -(x - a)$

This allows you to cancel the terms to get -1, e.g., $\dfrac{1 - x}{x - 1} = \dfrac{-(x - 1)}{x - 1} = -1$.

Drill B. Simplify the following to a single fraction with no additional fractions inside it.

(1) $\dfrac{\frac{1}{x} - \frac{1}{3}}{x - 3}$ (2) $\dfrac{\frac{1}{x + h} - \frac{1}{x}}{h}$ (3) $\dfrac{x^{-2} - y^{-2}}{x - y}$

Factoring out Powers

When we evaluate limits as $x \to \infty$, we need to create terms that approach zero (like $1/x$). We achieve this by "force factoring": factoring out the **highest power** of x, even if it isn't a common factor, to leave behind x's in the denominators.

Strategy: Factor-out the Highest Power. Factor out the **highest power** of x, forcing the remaining terms to have fractions with powers of x in the denominators.

I. Technique: $x^2 + 2x = x^2\left(1 + \dfrac{2x}{x^2}\right) = x^2\left(1 + \dfrac{2}{x}\right)$.

II. Why? As $x \to \infty$, the term $\dfrac{2}{x}$ vanishes to 0, simplifying the limit.

Drill C. Rewrite each expression by factoring out the **highest power** of x visible, ensuring remaining terms have x in the denominator where possible.

(1) $3x^3 - 5x + 10$

(2) $5x^4 + 2x$

(3) $\sqrt{x^4 - 7x^2}$

Two-step process: first factor out the highest power of x, and then separate the square roots; remember $\sqrt{AB} = \sqrt{A} \cdot \sqrt{B}$.

(4) $\dfrac{\sqrt{2x^2 + 3x}}{x}$

Drill D. Rewrite each expression by factoring out the **lowest power** of x visible and factor what remains.

(1) $3x^3 - 12x$

(2) $x^5 - 5x^4 + 6x^3$

(3) $2x^4 - 24x^3 + 70x^2$

also factor out a constant

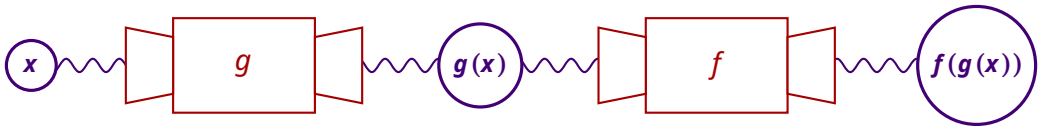
Algebra Skills Lab 3

Composition & Decomposition

The most powerful differentiation tool you will learn is the **Chain Rule**. To use it, you must intuitively grasp how functions are composed (chained together) and, more importantly, how to decompose them into "Inside" and "Outside" layers.

Visualising Composition

You may imagine composition as a relay race: the input x runs to function g, which hands off its result to function f.

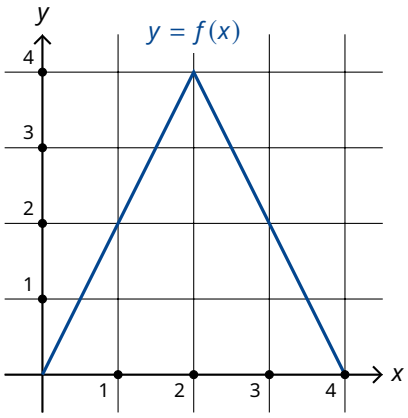


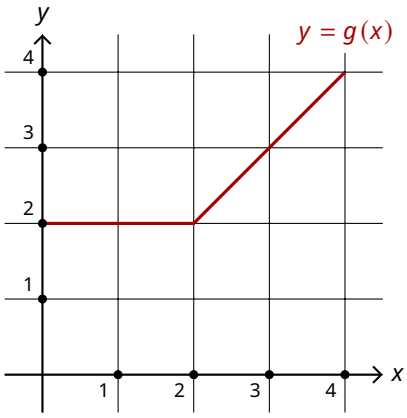
Strategy: The Two-Step Look Up. To evaluate $f(g(3))$ from graphs:

I. **Step 1. The Inside:** Find $g(3)$. Look at the graph of g at $x = 3$. Find the y-value.

II. **Step 2. The Outside:** Take that y-value and treat it as the ***x*-value** for the graph of f.

Drill A. Given are graphs of $f(x)$ and $g(x)$:





Use the graphs above to compute the following values. If a value does not exist, explain why.

(1) Find $g(3)$

(2) Find $f(g(3))$ using (1)

(3) Find $f(g(1))$

(4) Find $g(f(2))$

(5) Find $f(f(3))$

Decomposition (Identifying the Layers)

The Chain Rule requires you to identify the "inside" function (g) and the "outside" function (f):

$$h(x) = f(g(x)) \quad \text{means} \quad \text{Outside}(\text{Inside})$$

Strategy: The Onion Method. If you wish to evaluate evaluating the function $\ln(\sin(x))$ on a calculator. The **first** thing you type is the Inside ($g(x) = \sin$). The **last** operation you perform is the Outside ($f(x) = \ln$).

I. **Example:** $h(x) = \sin(x^2)$.

II. **Order:** First you square x; inside: $u = x^2$. Then you take sine; outside: $\sin(u)$.

Drill B. For each composite function $h(x)$, identify the **inside function** $g(x)$ and the **outside function** $f(x)$ such that

$$h(x) = f(g(x)).$$

Composite $h(x)$	**Inside $g(x)$**	**Outside $f(x)$**	**Check:** Does $f(g(x)) = h(x)$?
$(3x+1)^5$	$3x+1$	x^5	$(3x+1)^5$ ✔
$\sqrt{5x-2}$			
$\dfrac{1}{x^2+1}$			
$\cos(4x)$			
$\tan^2(x) = (\tan x)^2$			
$e^{\sqrt{x}}$			

Drill C. Sometimes there are three layers: $f(g(h(x)))$. Identify them for:

$$y = \sqrt{\sin(3x)}$$

- Innermost: _____
- Middle: _____
- Outermost: _____

Algebra Skills Lab 4

Synthesis: From Algebra to Calculus

We practised a few algebra skills: rationalising denominators, simplifying complex fractions, manipulating exponents, decomposing functions, etc.. In this lab, we will see exactly them put into action to solve some problems in Calculus.

The Limit Definition of the Derivative

The most important formula in the first half of Calculus is the definition of the derivative as a function in x:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If you try to plug in $h = 0$ immediately, you will always get $\frac{0}{0}$.

To find the answer, you must use algebra to simplify the fraction and cancel the h.

Strategy: Handling Square Roots. If $f(x)$ involves a square root, we will use the *conjugate method*.

I. **Set Up:** Write out $\frac{f(x+h) - f(x)}{h}$.

II. **Rationalise:** Multiply the top and bottom by the **conjugate** of the numerator.

III. **Simplify:** Obtain a *difference of squares* in the numerator. **Do not multiply out** the denominator.

IV. **Cancel:** Factor out h from the top and cancel it with the h on the bottom.

Drill A. Follow the steps to find the derivative of $f(x) = \sqrt{x+4}$ using the limit definition.

Step 1. Write the Difference Quotient $\frac{f(x+h) - f(x)}{h}$ using the given $f(x)$:

Step 2. Multiply numerator and denominator by the conjugate:

Step 3. Simplify the numerator; recall *difference of squares* $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$:

next page for the last step →

Step 4. Cancel the h from top and bottom. Then, set $h = 0$ to find the final answer.

Strategy: Handling Complex Fractions. If $f(x)$ involves a fraction, we will simplify the different quotient.

I. Set Up: Write out $\dfrac{f(x+h)-f(x)}{h}$.

II. Flip and Multiply: Rewrite "divide by h" as "multiply by $\dfrac{1}{h}$".

III. Combine: Combine the fractions that you have for $f(x+h)-f(x)$

IV. Cancel: The h should cancel.

Drill B. Find the derivative of $f(x) = \dfrac{1}{x+2}$ using the limit definition.

Step 1. Write the Difference Quotient $\dfrac{f(x+h)-f(x)}{h}$ using the given $f(x)$:

Step 2. Rewrite "divide by h" as "multiply by $\dfrac{1}{h}$" keeping the numerator as is:

Step 3. Combine the fractions in the numerator (find a common denominator between $x+h+2$ and $x+2$):

Step 4. Cancel the h. Set $h = 0$ to find the final answer.

Preparation for the Power Rule

We have learnt the **Power Rule**

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

For this, every term must be written strictly as a power of x. Roots and fractions must especially be re-written as powers.

Drill C. Rewrite each function as a sum of terms of the form $c \cdot x^n$, then apply the Power Rule.

Function y	**Rewrite** (Algebra)	**Derivative** y' (Calculus)
$\frac{1}{x^3} + \sqrt{x}$	$x^{-3} + x^{1/2}$	$-3x^{-4} + \frac{1}{2}x^{-1/2} = \frac{-3}{x^4} + \frac{1}{2\sqrt{x}}$
$\frac{4}{\sqrt[3]{x}}$		
$\frac{x^2 + 3x + 1}{x}$		
$x(3x^2 - 4)$		

Preparation for the Chain Rule

The **Chain Rule** states:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

To use this, you must successfully decompose the function into Outside (f) and Inside (g).

Drill D. For each function, identify the inside and outside, and then write the structure of the derivative.

Function $h(x)$	**Inside** $g(x)$	**Outside** $f(x)$	$g'(x)$	$f'(x)$	$f'(g(x))$	**Derivative** $h'(x) = f'(g(x)) \cdot g'(x)$
$(3x^2 + 1)^5$	$3x^2 + 1$	x^5	$6x$	$5x^4$	$5(3x^2 + 1)^4$	$5(3x^2 + 1)^4 \cdot (6x)$
$\sqrt{\sin(x)}$						
$e^{x^3 - 2x}$						
$\cos^4(x)$						

Limits at Infinity (End Behaviour)

To find horizontal asymptotes (limits as $x \to \infty$ or $x \to -\infty$), we use the technique of factoring out the **highest power** of x from the numerator *and* denominator.

Strategy: Dominant Powers. Factor out the **highest power** of x appearing in the numerator and denominator

I. For example, $x^2 + 2x = x^2\left(1 + \dfrac{2x}{x^2}\right) = x^2\left(1 + \dfrac{2}{x}\right)$.

II. **Why?** As $x \to \infty$ or $x \to -\infty$, any term like $\dfrac{c}{x^n}$ becomes 0.

III. **If you have a square root over a term in the numerator denominator:**

A. Do Step I to the expression under the square root and do not worry about the square root.

B. Re-write the expression as $\sqrt{x^n \cdot (\text{remaining})} = \sqrt{x^n} \cdot \sqrt{(\text{remaining})}$

C. If you obtain a $\sqrt{x^2}$, then $\sqrt{x^2} = x$ if your limit is $x \to \infty$ and $\sqrt{x^2} = -x$ if your limit is $x \to -\infty$

Drill E. Evaluate $\lim\limits_{x \to -\infty} \dfrac{\sqrt{9x^2 - 5x + 1}}{6x + 7}$ by following the steps.

Step 1. Factor the highest power of x in the numerator and denominator:

Step 2. Distribute the $\sqrt{\ }$ over the product in the numerator:

Step 3. Knowing that we are considering the limit as $x \to -\infty$, replace $\sqrt{x^2}$ appropriately

Step 4. Cancel whatever you can and replace terms of the form $\dfrac{c}{x^n}$ with 0 and compute the limit:

Algebra Skills Lab 5

Related Rates: Implicit Time & Geometry

This week in class we will discuss a concept known as related rates. In "Related Rates" problems, everything is changing with respect to **time** (t), which will be our independent variable.

This means every other variable (x, y, r, h, θ) is a dependent variable, i.e., in fact a function of time, e.g., $x(t)$. Consequently, whenever you take a derivative, you must use the Chain Rule.

Review: The "Hidden" Chain Rule in Implicit Differentiation

The Time Derivative. When differentiating x with respect to x, the answer is

$$x' = \frac{dx}{dx} = 1$$

However, when differentiating with respect to t, the derivative of x is

$$x' = \frac{dx}{dt}$$

Drill A. Differentiate the following equations with respect to **time** (t).

Don't forget the "prime" terms $\frac{dy}{dt}$, $\frac{dx}{dt}$, etc. attached to the end!

(1) $y = x^3$

(2) $x^2 + y^2 = 25$

(3) $\cos(\theta) = \frac{x}{10}$

(4) $V = 10\pi r^2$

Drill B. Differentiate the following with respect to t, using the Product or Quotient rules where necessary.

Don't forget that anything that is not t is a function in t so you must have "prime" terms attached to the end!

(1) $A = \frac{1}{2}bh$

(2) $\tan(\theta) = \frac{y}{x}$

Constants vs. Variables (Snapshot vs. Movie)

A confusing part of Related Rates is deciding what to plug in *before* the derivative and what to plug in *after*.

Strategy: The Movie Test. Imagine a movie of the problem happening.

- **Constant:** If a quantity **never changes** throughout the entire movie (e.g., the length of the ladder), you plug the number in **now** before differentiating. As a constant, its derivative is 0.
- **Variable:** If a quantity **changes** during the movie (e.g., the distance from or to the wall), you must leave it as a variable (for example, x). You plug the number in **later** after differentiating.

Drill C. A 10-foot ladder slides down a wall.
At a specific moment, the base is 6 feet from the wall and sliding away at 2 ft/sec.

(a) Draw a sketch of this scenario.

(b) Determine if the following quantities in this scenario is a **Constant** (C) or a **Variable** (V).

Quantity	**Type (C/V)**	**Value/Symbol to use**
The length of the ladder	C	10
The distance from base to wall		x
The height of the top of ladder		
The speed the ladder slides		

(c) Using (a), relate as many quantities as you can from (b) with an equation.

(d) Differentiate your equation from (c).

(e) Use (d) to compute the rate at which the ladder is sliding down the wall when the base is 6 feet away from the wall.

Drill D. Translate the English phrases into Calculus notation (using x, V, h, θ and derivatives).

(1) "The volume is increasing at a rate of 5 cubic cm per second." $\longrightarrow$ $\frac{dV}{dt} = 5$

(2) "The radius is decreasing at 2 cm per second." $\longrightarrow$

(3) "The water level (h) is rising at 4 cm/min." $\longrightarrow$

(4) "The distance (x) is unchanged." $\longrightarrow$

Algebra Skills Lab 7

The Method of *u*-Substitution

Integration by Substitution (usually called **"*u*-sub"**) is Chain Rule in reverse.

It is an algebraic process of changing everything under the integral in terms of the variable x (or whatever may be the original variable) to everything in terms of the new variable u to simplify an integral.

$$\int_0^1 x^3 \sin(x^4) dx$$

To succeed, you must manage 2 + 1 things perfectly: the **differential** (dx), the **leftover variables**, and the **bounds**.

In this method, you will choose an "inside function" labelling it $u = g(x)$.

Part I. Computing the Differential (du)

You first compute du using $u = g(x)$

$$u = g(x) \rightsquigarrow du = g'(x)\, dx$$

and then express dx in terms of du.

Drill A. For the given choice of u, calculate du and then isolate dx.

Choice of *u*	**Compute $du = g'(x)dx$**	**Isolate dx**	**Rewrite the expression in terms of only u & du**
$u = 3x + 1$	$du = 3\, dx$	$dx = \frac{1}{3} du$	$\sqrt{3x+1}\, dx =$
$u = x^4$	$du = 4x^3\, dx$	$x^3\, dx = \frac{1}{4} du$	$x^3 \sin(x^4)\, dx =$
$u = 5x$			$\cos(5x)\, dx =$
$u = 1 - x^2$			$\frac{x}{1-x^2}\, dx =$
$u = \sqrt{x} = x^{1/2}$			$\frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx =$
$u = \cos(x)$			$\tan(x)\, dx =$

Part II. The "Leftover" Variables (back-substitution)

Sometimes, after substituting u and du, there is still an x remaining in the expression. For example, if we repeat the steps from Drill A for

$$x\sqrt{x+1}\,dx$$

we have

$$u = x+1 \quad \rightsquigarrow \quad du = dx \quad \rightsquigarrow \quad x\sqrt{x+1}\,dx = x\sqrt{u}\,du$$

But we want our final expression to be entirely in terms of u. You must solve your original u-equation for x to get rid of it.

Strategy: Solve for x. If $u = x+1$, then simple algebra tells us that $x = u-1$.

We can plug $(u-1)$ into any remaining x in the expression to obtain

$$x\sqrt{x+1}\,dx = x\sqrt{u}\,du = (u-1)\sqrt{u}\,du$$

Drill B. Use back-substitution to write the given expression entirely in terms of u

Choice of *u*	***du***	**Rewrite in terms of only *u* & *du***
$u =$	$du =$	$x(1-x)^{3/2}\,dx =$
$u =$	$du =$	$x^2(x^2-1)^{1/2}\,dx =$
$u =$	$du =$	$\dfrac{x^3}{1-x^2}\,dx =$
$u =$	$du =$	$e^{\sqrt{x}}\,dx =$
$u =$	$du =$	$(\sin(x))^3\,dx =$

SCRATCH WORK

this will be useful: $(\sin(x))^3 = (\sin(x))^2 \cdot \sin(x)$ and $(\sin(x))^2 = 1-(\cos(x))^2$

Part III. Changing the Bounds (for definite integrals)

Warning. NEVER write *x*-bounds on a *u*-integral.

If the integral is $\int_a^b$, those numbers a and b are x-values ($x = a$ to $x = b$).

If you change variables to u, you **must** change the bounds to u-values.

$$\int_{x=a}^{x=b} (\text{function in terms of } x)\, dx = \int_{u=g(a)}^{u=g(b)} (\text{function in terms of } u)\, du$$

Drill C. Calculate the new Upper and Lower bounds in terms of u

Integral	Substitution with u	Old Bounds (x)	New Bounds (u)
$\int_0^2 2x(1-x^2)^3\, dx$	$u = 1 - x^2$	$x = 2$ $x = 0$	$u = 1 - (2)^2 = \mathbf{-3}$ $u = 1 - (0)^2 = \mathbf{1}$
$\int_1^4 \frac{1}{\sqrt{x} \cdot (1+\sqrt{x})^2}\, dx$	$u =$		
$\int_0^{\pi/2} \sin^3(x)\cos(x)\, dx$	$u =$		
$\int_{-1}^0 x(x+1)^5\, dx$	$u =$		

Rewrite these integrals in terms of u and du and with the new u-bounds

New Integral

_____

_____

_____

_____

Algebra Skills Lab 8

Review: Related Rates & Substitution

This lab synthesises the two major algebraic procedures we have learned recently:

1. **Related Rates:** Distinguishing between variables (the "Movie") and constants (the "Snapshot").
2. **Integration:** The distinct workflows for Indefinite vs. Definite integrals when using u-substitution.

Part I: The Movie vs. The Snapshot

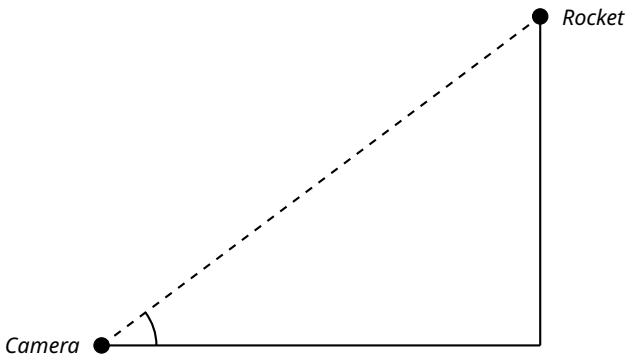
In Related Rates, a common error is plugging in numbers too early. **You must distinguish between:**

- **the equation that is true for *all time* (The Movie)**; and
- **the numbers that are true only at a *single moment* (The Snapshot).**

The Golden Rule of Related Rates.

- **The Movie (Differentiation Phase):** Use **Variables** (for example, x, y, θ).
  If a quantity changes over time, keep it as a letter. You differentiate this equation.
- **The Snapshot (Evaluation Phase):** Use **Values** (for example, $x = 4$, $y = 3$).
  Only plug these in *after* you have taken the derivative.

Drill A. Scenario: A camera on the ground is filming a rocket launching vertically. The camera is fixed 3000 ft from the launchpad. The rocket is rising.



(a) Label the moving quantities with variables and the constant with their values in the above diagram.

(b) Write three "Movie Equations" where each equation relates two unknown moving variables.

(c) Differentiate your Movie Equations with respect to t.

(d) You are told the rocket is rising with constant speed 500 ft/s. Plug in this value into two of your "Movie Equations"

(e) Your snapshot information is: what is the rate of change of the unknown quantities when the rocket is 4000 ft high.

Part II: The Fork in the Road (Definite vs. Indefinite)

When you perform a u-substitution, the computational path splits depending on whether you have

bounds – *definite integral* $\int_a^b$ or no bounds – *indefinite integral* $\int$

Two Different Endings with *u*-substitution.

I. Indefinite Integral $\int \cdots\, dx$: You want a function at the end.

- *Perform substitution.*
- *Integrate.*
- **Mandatory:** You **must** plug $u = g(x)$ back for your final answer.

II. Definite Integral $\int_a^b \cdots\, dx$: You want a number at the end.

- *Perform substitution.*
- **Mandatory:** You **must** calculate new bounds $u(a)$ and $u(b)$.
- *Integrate.*
- **Forbidden:** Do **not** go back to x at the end. Plug the new u-bounds into the u-antiderivative.

Drill B. Compare the process for the same integrand. **Do not just solve;** fill in the specific steps requested.

Let $u = x^2 + 1 \quad \rightsquigarrow \quad du = 2x\,dx$

Indefinite Integral $\int 2x(x^2+1)^3\,dx$	**Definite Integral** $\int_0^1 2x(x^2+1)^3\,dx$
1. Rewrite in terms of *u* $\int u^3\,du$	**1. Rewrite in terms of *u*** $\int_{u=?}^{u=?} u^3\,du$ **New Bounds:** $u(0) =$ $u(1) =$
2. Integrate (anti-differentiate) $\frac{u^4}{4} + C$	**2. Integrate (anti-differentiate)** $\left[\frac{u^4}{4}\right]_{\cdots}^{\cdots}$
3. Finish the problem	**3. Finish the problem**

Part III: Advanced Back-Substitution

Sometimes du does not remove every x. You must solve for x in the u-equation to finish the job.

Drill C. Evaluate

$$\int x^3\sqrt{x^2-3}\,dx$$

Step 1. Let $u = x^2 - 3$. Find du.

Step 2. Substitute back everything.

Have you eliminated all appearances of x? If not, then this is **illegal** and must be fixed.

Step 3. **The Fix:** Use $u = x^2 - 3$ to find the remaining x-term in terms of u and substitute this into the integral.

Step 4. Distribute and Integrate.

Step 5. **Finish the question.**

Algebra Skills Lab 9

Critical Points & Optimization

To find the maximum or minimum of a function, we must find where the slope is zero ($f'(x) = 0$). While the derivative rules are straightforward, solving the resulting equation often requires advanced factoring techniques, particularly with fractional exponents and exponentials.

Part I: Solving $f'(x) = 0$

Strategy: Solving Equations with Fractional Powers. When you apply the Power Rule to terms like $x^{1/2}$ or use the Product Rule, you often get expressions with negative exponents. To solve $f'(x) = 0$, do the following:

A. Example: Solve $x^{2/3} - x^{-1/3} = 0$.

B. Rewrite the Equation with Radicals & Fractions: $\sqrt[3]{x^2} + \dfrac{1}{\sqrt[3]{x}} = 0$

C. If the equation is undefined for some x, record that x-value: $\sqrt[3]{x^2} + \dfrac{1}{\sqrt[3]{x}}$ undefined at $x = 0$.

D. To solve multiply the equation with whatever appears in the denominator:

Multiply the equation $\sqrt[3]{x^2} + \dfrac{1}{\sqrt[3]{x}} = 0$ with $\sqrt[3]{x}$ giving us

$$\sqrt[3]{x^3} + 1 = 0$$

E. Simplify if you can and solve: $\sqrt[3]{x^3} = x$ and therefore the equation is $x + 1 = 0$ giving us $x = -1$.

Warning. Never cancel x. If you can cancel an x, then first record $x = 0$ as a possible solution first.

Drill A. Solve for x in the following equations (arising from derivatives).

(a) $3x^2 = 12x$

(b) $x^2e^x + 2xe^x = 0$ *Hint: Factor out xe^x. Recall e^x is never 0.*

(c) $1 - \dfrac{9}{x^2} = 0$

(d) $\frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = 0$ *Follow the strategy above.*

(e) $x - \frac{8}{x^2} = 0$

(f) $\frac{x^2 - 5x + 6}{x^2 + 4x + 4} = 0$

Drill B. In the equations in Drill A, record for each part if there was an x-value at which the expression was undefined.

Part II: Optimization (Setting Up the Model)

In Optimization problems, you are often given two variables (e.g., x and y) but you can only take the derivative of a function with **one** variable.

You must use a **constraint** equation to substitute one variable for the other.

Strategy: Constraint vs. Objective.

I. **Constraint:** The equation involving a **fixed number** given in the problem (e.g., "amount of fencing is 1200 ft"). Use this to solve for y.

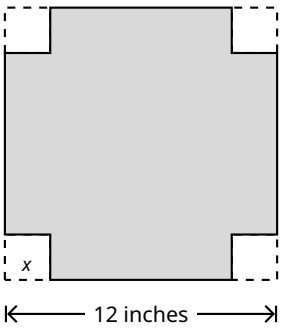
II. **Objective:** The equation you want to **maximise or minimise** (e.g., "maximise the area"). Substitute y into this equation.

Drill C. For each scenario, identify the Constraint and Objective, then produce the final single-variable function $f(x)$.

Scenario	**Constraint** (Fixed)	**Objective** (Max/Min)	**Function** $f(x)$
A farmer has 2400 ft of fencing (P) to enclose a rectangular field (A).	$2x + 2y = 2400$	$A = xy$	$A(x) = x(1200 - x)$
You need to construct a box with a square base and volume 32 ft^3. Minimize Surface Area.		$S = b^2 + 4bh$	
Product of two numbers is 100. Minimize their sum.			

Part III: Full Optimization Practice

Drill D. The Open Box Problem. A square sheet of cardboard 12 inches on a side is to be made into an open box by cutting squares of equal size (x) from each corner and folding up the flaps.



Step 1. Setup: Write the dimensions of the folded box in terms of x.

- Length = _____
- Width = _____
- Height = _____

Step 2. Objective Function: Write the Volume $V(x) = L \cdot W \cdot H$.

Step 3. Critical Points: Find $V'(x)$ and solve $V'(x) = 0$.

Step 4. Conclusion: Which x gives the maximum volume? (make sure to check the domain of V).

Algebra Skills Lab 10
A Self-Diagnostic

Aa tool for you to **audit** your own algebra toolkit before the final exam.

Calculus is 25% concept and 75% algebra; use this sheet to identify exactly which 90% you need to review.

Skill Inventory

Go through the list below. Be honest.

- **Green (G):** "I can do this in my sleep."
- **Yellow (Y):** "I can do this, but I usually need a hint or I make small arithmetic errors."
- **Red (R):** "I dread seeing this on a page. I don't know where to start."

The Algebra Skill	Lab Ref	G	Y	R
Pre-Calculus Foundations				
Rewriting roots as fractional exponents ($\sqrt[3]{x^2} \to x^{2/3}$)	Lab 1, 4	☐	☐	☐
Rewriting fractions as negative exponents ($1/x^3 \to x^{-3}$)	Lab 1, 4	☐	☐	☐
Expanding $(x+h)^2$ or $(x+h)^3$ correctly	Lab 1	☐	☐	☐
Limits & Derivatives				
Rationalising Numerators (Using Conjugates $\sqrt{A}+\sqrt{B}$)	Lab 2, 4	☐	☐	☐
Simplifying Complex Fractions (Common Denominators)	Lab 2, 4	☐	☐	☐
Factoring out Dominant Powers (Limits at ∞)	Lab 2, 4	☐	☐	☐
Decomposing Functions (Identifying Inner vs Outer)	Lab 3	☐	☐	☐
Solving $f'(x) = 0$ with fractional powers (e.g., $x^{-1/3}$)	Lab 9	☐	☐	☐
Applications (Rates & Optimization)				
Implicit Differentiation with respect to time ($x \rightsquigarrow dx/dt$)	Lab 5	☐	☐	☐
Distinguishing Constants vs. Variables (The Movie Test)	Lab 5, 8	☐	☐	☐
Identify Constraints vs. Objective Function	Lab 9	☐	☐	☐
Integration				
Computing Differential du and isolating dx	Lab 7	☐	☐	☐
Back-Substitution (Solving for x in terms of u)	Lab 7, 8	☐	☐	☐
Changing Bounds for Definite Integrals (x-bounds $\rightsquigarrow$ u-bounds)	Lab 7, 8	☐	☐	☐

Strategy Association

In a test, no one tells you "Use the conjugate method". You must recognise the visual cue yourself.

Draw a line connecting the **Visual Cue** to the correct **Algebraic Strategy**.

Visual Cue (The Problem)	Algebraic Strategy (The Tool)
$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}}{2x+5}$	**Multiply by Conjugate** $\frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}+\sqrt{B}}$
$\int x\sqrt{x+1}\,dx$	**Product Rule** First·d(Second) + Second·d(First)
$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$	**Implicit Differentiation** Do not forget $\frac{dx}{dt}$
Rate of change of Area $A = xy$	**Factor Dominant Power** e.g., divide all terms by x^2
Two cars moving apart $x^2 + y^2 = z^2$	**Back-Substitution** solve $u = x + 1 \rightsquigarrow x = u - 1$

Part III: Pitfall Patrol

These are the most common ways to lose points even if you know the calculus.

Check the box if you commit to **not** doing these things.

☐ **The Square Root error:** I will remember that $\sqrt{x^2+y^2} \neq x + y$.

☐ **The Power Rule error:** I will rewrite $\frac{1}{x^3}$ as x^{-3} before differentiating (not $\ln(x^3)$!).

☐ **The "Too Early" error:** In Related Rates, I will not plug in numbers until *after* I have differentiated.

☐ **The Bounds error:** If I write $\int_a^b f(u)\,du$, I promise that a and b are u-values, not x-values.

☐ **The Cancellation error:** I will not cancel x's in $\frac{x^2+1}{x}$ to get $x + 1$. (Splitting the fraction is okay!)

Final Game Plan

Look at your “Red” items in **Skill Inventory**.

Identify the two concepts that are most likely to appear on the Final Exam that you are currently *least* prepared for.

1. **Priority Concept 1:** _____

   I will fix this by re-doing **Drill** _____ from **Algebra Skills Lab #** _____

2. **Priority Concept 2:** _____

   I will fix this by re-doing **Drill** _____ from **Algebra Skills Lab #** _____